Logic at
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Abstract

These are notes for a part of the Logic Module taught at the ACM India Winter School 2026, *Joy of Problem Solving*, hosted at IIT Madras.

Chapter 1 is a pre-reading on the syntax and semantics of propositional logic, to be read before the first lecture.

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Propositional Logic

"Not the same thing a bit!" said the Hatter. "You might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"
Lewis Carroll, *Alice's Adventures in Wonderland*

In this chapter, we introduce propositional logic. We define its syntax, which says how formulas are built from atomic propositions using connectives, and its semantics, which says when a formula is true under a valuation. We end with satisfiability and logical consequence, the two notions used in the rest of these notes.

1.1 *Propositions*

A *proposition* is a statement that is either true or false. For example, "91 is prime" is a proposition, and a false one, since $91 = 7 \times 13$. Neither "Close the door" nor "$x > 3$" is a proposition: the first has no truth value, and the truth value of the second depends on x. Propositions combine into larger ones using words such as *not*, *or*, *and* and *if ... then*, and the truth value of a combination depends only on the truth values of its parts. Propositional logic studies what can be concluded from this structure alone. George Boole treated propositions algebraically in 1847 and 1854.[1]

[1] G. Boole, *The Mathematical Analysis of Logic*, Macmillan, Barclay and Macmillan, Cambridge, 1847; G. Boole, *An Investigation of the Laws of Thought*, Walton and Maberly, London, 1854. Boole wrote xy for "x and y" and worked with the equation $x^2 = x$, which holds for the truth values 0 and 1; Boolean algebra is named after him.

A puzzle On an island every inhabitant is either a *knight*, who always tells the truth, or a *knave*, who always lies.[2] Of two inhabitants, A says "B is a knave" and B says "A and I are of the same kind". Which of them is a knight? The answer is worked out in Section 1.4.

1.2 Syntax

Fix a nonempty, finite or countably infinite set $\mathscr{P} = \{p_1, p_2, \dots\}$ of *atomic propositions* (*atoms* for short), listed without repetitions; the letters p, q, r also denote atoms. There are two *logical connectives*, $\neg$ (read as *not*) and $\vee$ (read as *or*).[3] Formulas are strings over $\mathscr{P} \cup \{\neg, \vee, (,)\}$, where each atomic proposition counts as a single symbol, different from $\neg$, $\vee$, (and). The Greek letters α, β, γ stand for arbitrary formulas; they are not symbols of the language.

Definition 1.2.1 The set Φ of *formulas* is the smallest set of strings satisfying the following conditions, that is, the intersection of all sets of strings that satisfy them.

- Every atomic proposition p is a member of Φ.
- If α is a member of Φ, so is $(\neg\alpha)$.
- If α and β are members of Φ, so is $(\alpha \vee \beta)$.

For example, $((\neg(p \vee q)) \vee (\neg r))$ is a formula, built from p, q, r by four applications of the second and third rules. We omit the outermost parentheses, and $\neg$ binds more tightly than $\vee$, so this formula is written $\neg(p \vee q) \vee \neg r$.

Structural induction principle Since Φ is the smallest such set, it is a subset of every set satisfying the three conditions. This gives a method of proof. Let S be a set such that every atomic proposition is in S, $(\neg\alpha) \in S$ whenever $\alpha \in S$, and $(\alpha \vee \beta) \in S$ whenever $\alpha, \beta \in S$. Then $\Phi \subseteq S$.

To prove that every formula has a property, we take S to be the set of strings with that property. We then check that every atom has it, and that $(\neg\alpha)$ and $(\alpha \vee \beta)$ have it whenever α and β do.

Unique readability Because of the parentheses in the rules, each formula is built in only one way. Without them, the string $\neg p \vee q$ could be built either as the negation of $p \vee q$ or as the disjunction of $\neg p$ and q.

Proposition 1.2.2 (Unique readability) *Every formula is of exactly one of the forms p, $(\neg\alpha)$ and $(\alpha \vee \beta)$, and the formulas α and β are uniquely determined.*

---

[2] Puzzles of this kind were made popular by Raymond Smullyan, starting with R. M. Smullyan, *What Is the Name of This Book? The Riddle of Dracula and Other Logical Puzzles*, Prentice-Hall, Englewood Cliffs, 1978.

[3] The notation of these notes follows M. Mukund and S. P. Suresh, *An Introduction to Logic*, lecture notes, Chennai Mathematical Institute, 2011.

We omit the proof of this proposition. By Proposition 1.2.2, each formula can be drawn in exactly one way as a *parse tree*, with atomic propositions at the leaves and connectives at the inner nodes, as in Figure 1.1.

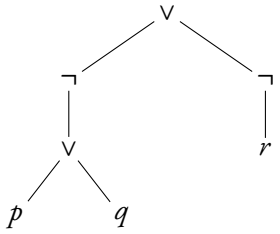


Figure 1.1 The parse tree of $\neg(p \vee q) \vee \neg r$, that is, of $((\neg(p \vee q)) \vee (\neg r))$.

Unique readability also lets us define functions on formulas by recursion. Suppose we give a value for each atomic proposition, and say how the values on $\neg\beta$ and $\beta \vee \gamma$ are computed from those on β and γ. Then every formula receives exactly one value. For example, the *vocabulary* $Voc(\alpha)$, the set of atomic propositions in α, is given by $Voc(p) = \{p\}$, $Voc(\neg\beta) = Voc(\beta)$ and $Voc(\beta \vee \gamma) = Voc(\beta) \cup Voc(\gamma)$. By structural induction, $Voc(\alpha)$ is always finite, since $\{p\}$ is finite and the union of two finite sets is finite.

Derived connectives We define three further connectives as abbreviations.

$$\alpha \wedge \beta \stackrel{\text{def}}{=} \neg(\neg\alpha \vee \neg\beta)$$
$$\alpha \supset \beta \stackrel{\text{def}}{=} \neg\alpha \vee \beta$$
$$\alpha \equiv \beta \stackrel{\text{def}}{=} (\alpha \supset \beta) \wedge (\beta \supset \alpha)$$

The connective $\wedge$ is read as *and*, $\supset$ as *implies*[4] and $\equiv$ as *if and only if*. The connectives bind in the order $\neg$, $\wedge$, $\vee$, $\supset$, $\equiv$, from tightest to loosest; $\wedge$ and $\vee$ group to the left, $\supset$ groups to the right, and repeated $\equiv$ is always parenthesised. For example, $p \wedge q \vee r \supset s \supset q$ means $((p \wedge q) \vee r) \supset (s \supset q)$. Since $\wedge$, $\supset$ and $\equiv$ are abbreviations, proofs by structural induction still need only the cases p, $\neg\alpha$ and $\alpha \vee \beta$.

1.3 Semantics

Let $\top$ denote the truth value true and $\bot$ the truth value false.

Definition 1.3.1 A *valuation* (or *interpretation*) is a function $v : \mathscr{P} \to \{\top, \bot\}$.

We extend each valuation v to a map $\hat{v} : \Phi \to \{\top, \bot\}$ by recursion.

- For $p \in \mathscr{P}$, $\hat{v}(p) = v(p)$.
- $\hat{v}(\neg\beta) = \top$ if $\hat{v}(\beta) = \bot$, and $\bot$ otherwise.

[4] Giuseppe Peano wrote a reversed letter C for "implies" in *Arithmetices principia, nova methodo exposita* (1889), and Whitehead and Russell used $\supset$ in *Principia Mathematica* (1910–1913). Many books now write $\to$ instead.

- $\hat{v}(\beta \vee \gamma) = \bot$ if $\hat{v}(\beta) = \hat{v}(\gamma) = \bot$, and $\top$ otherwise.

By unique readability, $\hat{v}$ is well defined. We write v for $\hat{v}$, and $v \vDash \alpha$ ("v satisfies α") when $v(\alpha) = \top$; otherwise we write $v \nvDash \alpha$.

Proposition 1.3.2 *For all formulas α, β and every valuation v, the connectives take the following values.*

$v(\alpha)$	$v(\beta)$	$v(\neg\alpha)$	$v(\alpha \vee \beta)$	$v(\alpha \wedge \beta)$	$v(\alpha \supset \beta)$	$v(\alpha \equiv \beta)$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$

Proof. For $\wedge$: $v(\neg(\neg\alpha \vee \neg\beta)) = \top$ if and only if $v(\neg\alpha) = v(\neg\beta) = \bot$, that is, $v(\alpha) = v(\beta) = \top$. For $\supset$: $v(\neg\alpha \vee \beta) = \bot$ if and only if $v(\alpha) = \top$ and $v(\beta) = \bot$. Since $\alpha \equiv \beta$ abbreviates $(\alpha \supset \beta) \wedge (\beta \supset \alpha)$, it is true exactly when both implications are, that is, when $v(\alpha) = v(\beta)$. The columns for $\neg$ and $\vee$ are the definition of v.

The disjunction $\vee$ is inclusive: $\alpha \vee \beta$ is true when both parts are. The implication $\alpha \supset \beta$ is false only when α is true and β false. For example, "if you score above 90, you get an A" is false only for a student who scores above 90 and does not get an A. For a student who scores 90 or below, the statement is true whatever grade they get. In general, $\alpha \supset \beta$ is true whenever α is false.

Proposition 1.3.3 *If valuations v_1 and v_2 agree on $Voc(\alpha)$, then $v_1(\alpha) = v_2(\alpha)$.*

Proof. By structural induction on α. For $\alpha = p$ this is the assumption. If $\alpha = \neg\beta$, then $Voc(\beta) = Voc(\alpha)$, so $v_1(\beta) = v_2(\beta)$ by the induction hypothesis and hence $v_1(\alpha) = v_2(\alpha)$. If $\alpha = \beta \vee \gamma$, then v_1 and v_2 agree on $Voc(\beta)$ and on $Voc(\gamma)$, so $v_1(\beta) = v_2(\beta)$ and $v_1(\gamma) = v_2(\gamma)$ by the induction hypothesis, and hence $v_1(\alpha) = v_2(\alpha)$.

If $Voc(\alpha)$ has n elements, there are 2^n ways to give them truth values, and by Proposition 1.3.3 these determine $v(\alpha)$ for every v. A *truth table* lists the 2^n cases, one per row, with the value of α in each. Truth tables in this form were introduced in 1921, independently, by Emil Post and Ludwig Wittgenstein.[5] The following truth table for $((p \supset q) \wedge p) \supset q$ has a column for each subformula.

[5] E. L. Post, "Introduction to a general theory of elementary propositions", *American Journal of Mathematics* 43 (1921), 163–185; L. Wittgenstein, "Logisch-philosophische Abhandlung", *Annalen der Naturphilosophie* 14 (1921), 185–262, translated as *Tractatus Logico-Philosophicus*, 1922. Post also proved that every formula true under all valuations can be derived in the proof system of Whitehead and Russell's *Principia Mathematica*; this was the first completeness theorem to be published. Paul Bernays had proved the same result in 1918, in an unpublished Habilitationsschrift.

p	q	$p \supset q$	$(p \supset q) \wedge p$	$((p \supset q) \wedge p) \supset q$
$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\top$	$\bot$	$\top$

The last column is $\top$ in every row, so this formula is true under every valuation.

1.4 Satisfiability and consequence

A formula α is *satisfiable* if $v \vDash \alpha$ for some valuation v. For example, $p \supset q$ is satisfiable and $p \wedge \neg p$ is not. By Proposition 1.3.3, a truth table decides whether a formula is satisfiable: it is satisfiable if and only if some row gives $\top$.

Consequence Let X be a set of formulas. A formula α is a *logical consequence* of X, written $X \vDash \alpha$, if every valuation that satisfies all formulas in X satisfies α; otherwise we write $X \nvDash \alpha$. In the truth table at the end of Section 1.3, only the first row makes $p \supset q$ and p true, and there q is true, so $\{p \supset q,\ p\} \vDash q$. The third row makes $p \supset q$ and q true and p false, so $\{p \supset q,\ q\} \nvDash p$.

***Exercise** 1.4.1* Let $n \geq 1$. Prove that $\{\alpha_1, \ldots, \alpha_n\} \vDash \beta$ if and only if $\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \neg\beta$ is not satisfiable.

The puzzle, solved Let a stand for "A is a knight" and b for "B is a knight". A speaker is a knight exactly when what they say is true, so the two statements become $a \equiv \neg b$ and $b \equiv (a \equiv b)$. A solution is a valuation satisfying both.

a	b	$a \equiv \neg b$	$a \equiv b$	$b \equiv (a \equiv b)$
$\top$	$\top$	$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\bot$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$

Only the second row satisfies both formulas, so the puzzle has exactly one solution: A is a knight and B is a knave.

***Exercise** 1.4.2* Another inhabitant, C, says to an inhabitant D: "At least one of us is a knave." Is each of C and D a knight or a knave?

***Exercise** 1.4.3* Exactly one of Asha, Bilal and Chitra broke the window. Asha says "Bilal did it." Bilal says "I did not." Chitra says "I did not." Exactly one of the three statements is true. Write these facts as formulas and find who broke the window. Let a, b, c stand for "Asha broke it", "Bilal broke it" and "Chitra broke it". The statement that exactly one of α, β, γ is true can be written as $(\alpha \wedge \neg\beta \wedge \neg\gamma) \vee (\neg\alpha \wedge \beta \wedge \neg\gamma) \vee (\neg\alpha \wedge \neg\beta \wedge \gamma)$.

Compactness

Mathematics is the science of the infinite, its goal the symbolic comprehension of the infinite with human, that is finite, means.
Hermann Weyl, *The Open World*

In this chapter, we prove the compactness theorem for propositional logic: a set of formulas, possibly infinite, is satisfiable if every finite subset of it is satisfiable. The proof extends a finitely satisfiable set to a maximal one and reads a valuation off the maximal set. We then state the theorem in terms of logical consequence, and end with the topological fact that gives the theorem its name.

2.1 Satisfiable sets of formulas

Chapter 1 asked whether a single formula is satisfiable. Here we ask the same question of a set of formulas, which may be infinite.

Definition 2.1.1 Let $X \subseteq \Phi$. A valuation v *satisfies* X, written $v \vDash X$, if $v \vDash \beta$ for every $\beta \in X$. The set X is *satisfiable* if some valuation satisfies it. We write $Y \subseteq_{\text{fin}} X$ when Y is a finite subset of X, and call X *finitely satisfiable* if every $Y \subseteq_{\text{fin}} X$ is satisfiable.

A finite nonempty set $X = \{\beta_1, \dots, \beta_n\}$ is satisfiable if and only if the single formula $\beta_1 \wedge \dots \wedge \beta_n$ is, so a truth table decides the question. An infinite set X may mention infinitely many atomic propositions, and then no truth table is available. Every finite subset of X may be satisfiable, each by its own valuation, and it is not clear that these

valuations can be combined into one. The compactness theorem says that a single valuation satisfying all of X always exists.

Theorem 2.1.2 (Compactness) *Let $X \subseteq \Phi$. Then X is satisfiable if and only if it is finitely satisfiable.*

If $v \vDash X$ and $Y \subseteq X$, then $v \vDash \beta$ for every $\beta \in Y$, so $v \vDash Y$. Hence a satisfiable set is finitely satisfiable. The converse is proved in Section 2.3, using the lemmas of Section 2.2.

In 1930 Kurt Gödel proved compactness for countable languages as a consequence of his completeness theorem for first-order logic, and the propositional version is a special case of his result.[1] Anatoly Mal'cev proved it for languages of any size in 1936, and later used it to prove theorems about infinite groups.[2] The proof below works directly with satisfiability and needs no proof system.

***Exercise** 2.1.3* Let $\mathscr{P}$ be infinite and $X = \{p_1\} \cup \{p_i \supset p_{i+1} : i \geq 1\}$. Show that X is satisfiable and that $X \cup \{\neg p_5\}$ is not. Find a finite subset of $X \cup \{\neg p_5\}$ that is not satisfiable.

2.2 *Maximal finitely satisfiable sets*

To find a valuation for a finitely satisfiable set X, we first enlarge X as far as possible while keeping it finitely satisfiable, and then read the valuation off the larger set.

Definition 2.2.1 A finitely satisfiable set $\Delta \subseteq \Phi$ is *maximal* if for every formula $\alpha \notin \Delta$, the set $\Delta \cup \{\alpha\}$ is not finitely satisfiable.

Example 2.2.2 For a valuation v, the set $\Delta_v = \{\alpha : v \vDash \alpha\}$ is maximal finitely satisfiable. Every finite subset of Δ_v is satisfied by v, so Δ_v is finitely satisfiable. If $\alpha \notin \Delta_v$, then $v \vDash \neg\alpha$, so $\neg\alpha \in \Delta_v$, and the finite set $\{\alpha, \neg\alpha\}$ shows that $\Delta_v \cup \{\alpha\}$ is not finitely satisfiable.

The proof of compactness goes in the opposite direction, from a maximal finitely satisfiable set to a valuation.

Lemma 2.2.3 *If Z is finitely satisfiable and α is a formula, then $Z \cup \{\alpha\}$ or $Z \cup \{\neg\alpha\}$ is finitely satisfiable.*

Proof. Suppose neither is. Since Z is finitely satisfiable, a finite subset of $Z \cup \{\alpha\}$ that is not satisfiable must contain α, and similarly for $\neg\alpha$. So there are finite $Y_1, Y_2 \subseteq Z$ such that $Y_1 \cup \{\alpha\}$ and $Y_2 \cup \{\neg\alpha\}$ are not satisfiable. The finite set $Y_1 \cup Y_2 \subseteq Z$ is satisfied by some valuation v. If $v(\alpha) = \top$, then $v \vDash Y_1 \cup \{\alpha\}$; if $v(\alpha) = \bot$, then $v \vDash Y_2 \cup \{\neg\alpha\}$. Either way we have a contradiction.

[1] K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", *Monatshefte für Mathematik und Physik* 37 (1930), 349–360. A proof uses only finitely many premises, so if every finite subset of X is consistent, then so is X; completeness turns consistency into satisfiability.

[2] A. Mal'cev (published as A. Malcev), "Untersuchungen aus dem Gebiete der mathematischen Logik", *Matematicheskii Sbornik* 1(43) (1936), 323–336.

The next lemma is due to Adolf Lindenbaum, a Polish logician killed in 1941, during the Second World War; it was first published by Alfred Tarski.[3]

Lemma 2.2.4 (Lindenbaum) *Every finitely satisfiable set X is contained in a maximal finitely satisfiable set.*

Proof. The set Φ is countable: its alphabet $\mathscr{P} \cup \{\neg, \vee, (,)\}$ is countable, and for each length k there are countably many strings of length k. So the set of all strings is a countable union of countable sets, which is countable. Since Φ is a set of strings, it is countable too. List the formulas as $\alpha_1, \alpha_2, \ldots$. Let $\Delta_0 = X$, and let

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\alpha_{n+1}\} & \text{if this set is finitely satisfiable,} \\ \Delta_n \cup \{\neg\alpha_{n+1}\} & \text{otherwise.} \end{cases}$$

By Lemma 2.2.3 and induction on n, every Δ_n is finitely satisfiable. Let $\Delta = \bigcup_n \Delta_n$ (Figure 2.1). Let $Y \subseteq \Delta$ be finite. Since $\Delta_0 \subseteq \Delta_1 \subseteq \cdots$, the set Y lies in some Δ_n, so Y is satisfiable. Hence Δ is finitely satisfiable. At each step n one of α_n and $\neg\alpha_n$ was added, so one of them lies in Δ. So if $\alpha \notin \Delta$, then $\neg\alpha \in \Delta$, and $\Delta \cup \{\alpha\}$ contains the set $\{\alpha, \neg\alpha\}$, which is not satisfiable. Thus Δ is maximal.

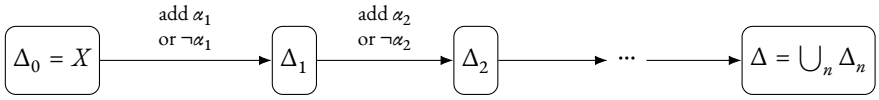


Figure 2.1 The construction of Δ. At step n we add α_n if this keeps the set finitely satisfiable, and $\neg\alpha_n$ otherwise.

Lemma 2.2.5 *Let Δ be maximal finitely satisfiable. For all formulas α and β:*

(a) exactly one of α and $\neg\alpha$ is in Δ;

(b) $\alpha \vee \beta \in \Delta$ if and only if $\alpha \in \Delta$ or $\beta \in \Delta$.

Proof. (a) Not both, since $\{\alpha, \neg\alpha\}$ is not satisfiable. If neither were in Δ, then by maximality neither $\Delta \cup \{\alpha\}$ nor $\Delta \cup \{\neg\alpha\}$ would be finitely satisfiable, contradicting Lemma 2.2.3.

(b) Suppose $\alpha \vee \beta \in \Delta$ but $\alpha, \beta \notin \Delta$. By (a), $\neg\alpha, \neg\beta \in \Delta$, and the finite set $\{\alpha \vee \beta, \neg\alpha, \neg\beta\} \subseteq \Delta$ is not satisfiable, contradicting the finite satisfiability of Δ. Conversely, suppose $\alpha \in \Delta$ but $\alpha \vee \beta \notin \Delta$. By (a), $\neg(\alpha \vee \beta) \in \Delta$, and $\{\alpha, \neg(\alpha \vee \beta)\} \subseteq \Delta$ is not satisfiable, again a contradiction. The case $\beta \in \Delta$ is the same.

[3] A. Tarski, "Über einige fundamentale Begriffe der Metamathematik", *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Classe III* 23 (1930), 22–29; a fuller account is in A. Tarski, "Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften. I", *Monatshefte für Mathematik und Physik* 37 (1930), 361–404. Tarski stated it for consistent sets of sentences in a proof system; the version here uses finite satisfiability instead.

The valuation of a maximal set For a maximal finitely satisfiable set Δ, define the valuation v_Δ by $v_\Delta(p) = \top$ if and only if $p \in \Delta$.

Lemma 2.2.6 (Truth lemma) *Let Δ be maximal finitely satisfiable. For every formula α, $v_\Delta \models \alpha$ if and only if $\alpha \in \Delta$.*

Proof. By structural induction. For an atomic proposition this is the definition of v_Δ. For a negation $\neg\beta$,

$$\begin{aligned} v_\Delta \models \neg\beta &\iff v_\Delta \not\models \beta \\ &\iff \beta \notin \Delta && \text{(induction hypothesis)} \\ &\iff \neg\beta \in \Delta && \text{(Lemma 2.2.5(a)).} \end{aligned}$$

For a disjunction $\beta \vee \gamma$,

$$\begin{aligned} v_\Delta \models \beta \vee \gamma &\iff v_\Delta \models \beta \text{ or } v_\Delta \models \gamma \\ &\iff \beta \in \Delta \text{ or } \gamma \in \Delta && \text{(induction hypothesis)} \\ &\iff \beta \vee \gamma \in \Delta && \text{(Lemma 2.2.5(b)).} \end{aligned}$$

***Exercise** 2.2.7* Let Δ be maximal finitely satisfiable. Show that $\alpha \wedge \beta \in \Delta$ if and only if $\alpha \in \Delta$ and $\beta \in \Delta$, and that $\alpha \supset \beta \in \Delta$ if and only if $\alpha \notin \Delta$ or $\beta \in \Delta$.

2.3 Proof of the compactness theorem

Proof of Theorem 2.1.2. A satisfiable set is finitely satisfiable, as shown after the statement of the theorem. Conversely, let X be finitely satisfiable. By Lemma 2.2.4, X is contained in a maximal finitely satisfiable set Δ. By Lemma 2.2.6, $v_\Delta \models \Delta$, and so $v_\Delta \models X$ (Figure 2.2).

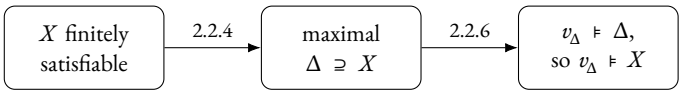


Figure 2.2 The steps in the proof of Theorem 2.1.2. Each arrow is labelled by the lemma it uses.

***Exercise** 2.3.1* Show that every maximal finitely satisfiable set is Δ_v for some valuation v, as in Example 2.2.2.

***Exercise** 2.3.2* Let $\mathcal{P}$ be infinite and $X = \{p_1\} \cup \{p_i \supset p_{i+1} : i \geq 1\}$. Show that exactly one maximal finitely satisfiable set contains X.

***Exercise** 2.3.3* (Kőnig's lemma) A rooted tree is *finitely branching* if every node has finitely many children. Use Theorem 2.1.2 to show that every infinite, finitely branching tree has an infinite path from the root.[4] (Each depth has finitely many nodes, and since the tree is infinite, every depth has at least one. Take an atom p_x for each node x, read as "x is on the path", and write formulas saying that exactly one node at each depth is on the path, and that the parent of a node on the path is on the path.)

***Exercise** 2.3.4* Where does the proof use that $\mathcal{P}$ is countable? Where does it use that formulas are finite strings?

2.4 Compactness and consequence

By definition (Section 1.4), $X \models \alpha$ means that every valuation satisfying X satisfies α. Compactness implies that a consequence of an infinite set follows from finitely many of its members. The proof uses the following lemma, which turns a question about consequence into one about satisfiability.

Lemma 2.4.1 *For all $Z \subseteq \Phi$ and $\beta \in \Phi$, $Z \models \beta$ if and only if $Z \cup \{\neg\beta\}$ is not satisfiable.*

Proof. A valuation v satisfies $Z \cup \{\neg\beta\}$ exactly when it satisfies Z and $v(\beta) = \bot$. Such a valuation exists if and only if $Z \not\models \beta$.

For finite nonempty Z this is equivalent to Exercise 1.4.1.

Theorem 2.4.2 *Let $X \subseteq \Phi$ and $\alpha \in \Phi$. Then $X \models \alpha$ if and only if $Y \models \alpha$ for some $Y \subseteq_{\text{fin}} X$.*

Proof. ($\Leftarrow$) Let $Y \subseteq_{\text{fin}} X$ with $Y \models \alpha$. If $v \models X$, then $v \models Y$, so $v \models \alpha$.

($\Rightarrow$) Suppose $X \models \alpha$. By Lemma 2.4.1, $X \cup \{\neg\alpha\}$ is not satisfiable. By Theorem 2.1.2, some $Z \subseteq_{\text{fin}} X \cup \{\neg\alpha\}$ is not satisfiable. Let $Y = Z \setminus \{\neg\alpha\}$, a finite subset of X. Then $Z \subseteq Y \cup \{\neg\alpha\}$, so a valuation satisfying $Y \cup \{\neg\alpha\}$ would satisfy Z. Hence $Y \cup \{\neg\alpha\}$ is not satisfiable, and $Y \models \alpha$ by Lemma 2.4.1.

2.5 A topological view

This section assumes topology and is beyond the course; it explains the name of the theorem. Marshall Stone, who connected Boolean algebras, and with them propositional logic, to compact spaces, wrote in 1938:[5]

> *A cardinal principle of modern mathematical research may be stated as a maxim: "One must always topologize."*

[4] D. Kőnig, "Über eine Schlussweise aus dem Endlichen ins Unendliche", *Acta Scientiarum Mathematicarum (Szeged)* 3 (1927), 121–130; the journal then had a longer Latin title. The article's title translates as "On a method of inference from the finite to the infinite", which describes compactness as well.

[5] M. H. Stone, "The representation of Boolean algebras", *Bulletin of the American Mathematical Society* 44 (1938), 807–816, an expository address. Stone had shown in 1936 that every Boolean algebra is isomorphic to the algebra of clopen subsets of a compact Hausdorff space, now called its Stone space; the space of valuations used here is a special case.

Give $\{\top, \bot\}$ the discrete topology, and give the set $\{\top, \bot\}^{\mathscr{P}}$ of all valuations the product topology. By Tychonoff's theorem this space is compact.[6] Each formula α determines the set $[\![\alpha]\!] = \{v : v \vDash \alpha\}$. The value $v(\alpha)$ depends only on the finitely many values $v(p)$ for $p \in \mathit{Voc}(\alpha)$, so $[\![\alpha]\!]$ is a finite union of basic open sets, and hence open. Its complement $[\![\neg\alpha]\!]$ is open for the same reason, so $[\![\alpha]\!]$ is also closed. A set X is satisfiable exactly when $\bigcap_{\alpha\in X}[\![\alpha]\!]$ is nonempty, and finitely satisfiable exactly when the family $\{[\![\alpha]\!] : \alpha \in X\}$ has the finite intersection property. In a compact space every family of closed sets with the finite intersection property has nonempty intersection. Applied to the sets $[\![\alpha]\!]$ for $\alpha \in X$, this property of compact spaces gives Theorem 2.1.2. The argument works for a set of atomic propositions of any size, though Tychonoff's theorem, on which it rests, uses the axiom of choice.

[6] A. Tychonoff, "Über die topologische Erweiterung von Räumen", *Mathematische Annalen* 102 (1930), 544–561. For products of finite spaces, as here, the theorem can also be proved with Kőnig's lemma when $\mathscr{P}$ is countable.

Applications I

Good mathematicians see analogies between theorems or theories; the very best ones see analogies between analogies.
Stefan Banach, as recalled by Stanisław Ulam

In this chapter, we apply the compactness theorem to three problems: colouring infinite graphs, matching in infinite bipartite graphs, and tiling the plane with Wang tiles. In each of the three cases, we assume the finite case and use compactness to pass to the infinite case. For colouring and matching the finite case is a known theorem; for tiling it is a hypothesis.

In each case we choose atomic propositions that describe the infinite object, such as r_u for "vertex u is red", and write down a set X of formulas such that a valuation satisfying X gives the object we want. A finite subset of X mentions only a finite part of the object, and the finite case shows that this subset is satisfiable. Compactness then gives a valuation satisfying all of X. Since Chapter 1 fixes a countable set of atomic propositions, each proof also checks that it uses only countably many of them.

3.1 Graph colouring

A *graph* $G = (V, E)$ consists of a set V of *vertices* and a set E of *edges*, each a set $\{u, w\}$ of two distinct vertices. For $k \geq 1$, a *k-colouring* of G is a function $c : V \to \{1, \dots, k\}$ with $c(u) \neq c(w)$ for every edge $\{u, w\}$. For $V_0 \subseteq V$, the *subgraph induced by V_0* has vertex set

V_0 and the edges of G with both ends in V_0. A finite graph is *planar* if it can be drawn in the plane without edges crossing.

Theorem 3.1.1 (Four colour theorem; Appel and Haken, 1976) *Every finite planar graph has a 4-colouring.*

Francis Guthrie posed the four colour problem in 1852, and Kenneth Appel and Wolfgang Haken proved the theorem in 1976.[1] We omit the proof, and ask whether the result holds for an infinite graph, such as the one in Figure 3.1. Colouring such a graph piece by piece may fail, since a colouring chosen for one finite piece may not extend to a larger piece. In the proof below, no colouring of a finite piece is fixed in advance.

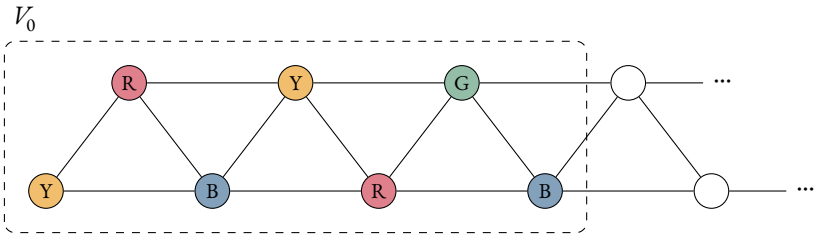


Figure 3.1 Part of an infinite planar graph, a strip of triangles that continues to the right. The finite set V_0 induces a planar subgraph, which has a 4-colouring (R, Y, B, G).

Theorem 3.1.2 *Let $G = (V, E)$ be a graph with V countable, and suppose every finite induced subgraph of G is planar. Then G has a 4-colouring.*

Proof. Take atomic propositions r_u, y_u, b_u, g_u for each $u \in V$, read as "u is red", "yellow", "blue", "green"; there are countably many of them. Let X consist of these formulas.

(a) For each $u \in V$: $r_u \vee y_u \vee b_u \vee g_u$.

(b) For each $u \in V$ and each pair of distinct colours: the formulas $\neg(r_u \wedge y_u)$, $\neg(r_u \wedge b_u)$, ..., $\neg(b_u \wedge g_u)$, one for each of the six pairs.

(c) For each edge $\{u, w\}$: $\neg(r_u \wedge r_w)$, $\neg(y_u \wedge y_w)$, $\neg(b_u \wedge b_w)$, $\neg(g_u \wedge g_w)$.

By (a) and (b), a valuation satisfying X makes exactly one colour atom true at each vertex. Reading red, yellow, blue and green as 1, 2, 3 and 4, the valuation defines a function $c : V \to \{1, 2, 3, 4\}$, and by (c), c is a 4-colouring.

By Theorem 2.1.2 it is enough to show that every $Y \subseteq_{\text{fin}} X$ is satisfiable. Let V_0 be the finite set of vertices whose atoms occur in Y. The subgraph induced by V_0 is planar, so it has a 4-colouring c_0. Let v be the valuation that makes the atom for colour $c_0(u)$ true at each $u \in V_0$ and every other atom false. Each formula of Y mentions only vertices in V_0.

[1] Guthrie's question reached Augustus De Morgan, who described it in a letter to William Rowan Hamilton dated 23 October 1852. The proof is K. Appel and W. Haken, "Every planar map is four colorable", *Bulletin of the American Mathematical Society* 82 (1976), 711–712. It reduces the problem to a large number of configurations checked by computer. N. Robertson, D. Sanders, P. Seymour and R. Thomas gave a simpler proof in 1997, still using a computer, and G. Gonthier checked a proof in the Coq proof assistant in 2005.

The valuation v makes exactly one colour atom true at each vertex of V_0, so the formulas of types (a) and (b) in Y hold. Each formula of type (c) in Y comes from an edge of the induced subgraph, which c_0 colours properly. So $v \models Y$.

***Exercise** 3.1.3* Write out the formulas of types (a), (b) and (c) for the graph with two vertices u, w and one edge $\{u, w\}$, and give a valuation that satisfies them all.

***Exercise** 3.1.4* Find a valuation that satisfies the formulas of types (b) and (c), whatever the graph, but does not define a colouring; so the formulas of type (a) are needed. Then show that the formulas of type (b) can be dropped, by obtaining a 4-colouring from a valuation that satisfies (a) and (c). (Hint: choose one true colour atom at each vertex.)

The de Bruijn–Erdős theorem Nothing in the proof used planarity except to colour each finite piece. The same argument proves the following theorem, with atoms $p_{u,i}$ for $u \in V$ and $1 \leq i \leq k$, read as "u has colour i". There are countably many such atoms, since $V \times \{1, \dots, k\}$ is countable.

Theorem 3.1.5 (de Bruijn and Erdős, 1951) *Let G be a graph with countably many vertices, and let $k \geq 1$. If every finite induced subgraph of G has a k-colouring, then so does G.*

Theorem 3.1.2 is the case $k = 4$, with the four colour theorem supplying the colourings of the finite subgraphs. Nicolaas de Bruijn and Paul Erdős published the theorem in 1951.[2] The de Bruijn–Erdős theorem also holds for graphs with uncountably many vertices, by the topological form of compactness in Section 2.5.

3.2 Hall's marriage theorem

Let B be a set of boys and $\mathcal{G}$ a set of girls, and suppose each boy b knows a set $N(b) \subseteq \mathcal{G}$ of girls. For $S \subseteq B$ let $N(S) = \bigcup_{b \in S} N(b)$. A *matching* is an injective function $f : B \to \mathcal{G}$ with $f(b) \in N(b)$ for every b: each boy marries a girl he knows, and no girl marries twice (Figure 3.2(a)). If a matching exists, every finite set S of boys knows at least $|S|$ girls, since f maps S injectively into $N(S)$. This is *Hall's condition*:

$$|N(S)| \geq |S| \quad \text{for every finite } S \subseteq B.$$

[2] N. G. de Bruijn and P. Erdős, "A colour problem for infinite graphs and a problem in the theory of relations", *Indagationes Mathematicae* 13 (1951), 371–373.

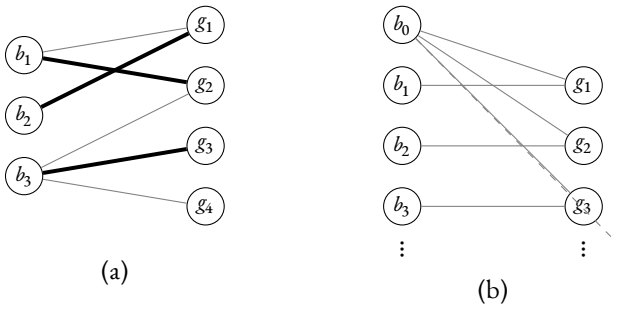


Figure 3.2 (a) Thin lines join each boy to the girls he knows; thick lines form a matching. (b) The boy b_0 knows every girl, and b_i knows only g_i. Hall's condition holds, but there is no matching (see Exercise 3.2.4).

***Exercise** 3.2.1* Check Hall's condition for the boys in Figure 3.2(a), and find a matching different from the one drawn.

Theorem 3.2.2 (P. Hall, 1935) *If B is finite, each $N(b)$ is finite, and Hall's condition holds, then a matching exists.*

We omit the proof.[3] Marshall Hall Jr. extended the theorem to infinitely many boys.[4]

Theorem 3.2.3 (M. Hall Jr., 1948) *Let B be countable, let each $N(b)$ be finite, and suppose Hall's condition holds. Then a matching exists.*

Proof. Take an atomic proposition $m_{b,g}$ for each $b \in B$ and $g \in N(b)$, read as "b marries g". There are countably many, since B is countable and each $N(b)$ is finite. Let X consist of the following formulas.

(a) For each b: the disjunction of $m_{b,g}$ over $g \in N(b)$. This is a finite disjunction, and $N(b)$ is nonempty by Hall's condition for $S = \{b\}$.

(b) For each b and distinct $g, g' \in N(b)$: $\neg(m_{b,g} \wedge m_{b,g'})$.

(c) For each g and distinct b, b' with $g \in N(b) \cap N(b')$: $\neg(m_{b,g} \wedge m_{b',g})$.

By (a) and (b), a valuation satisfying X makes $m_{b,g}$ true for exactly one g, for each b; let $f(b)$ be that g. By (c), f is injective.

Let $Y \subseteq_{\text{fin}} X$, and let B_0 be the finite set of boys whose atoms occur in Y. Every subset of B_0 is a finite subset of B, so Hall's condition holds for the boys in B_0, and the finite theorem gives a matching f_0 of B_0. Let v make $m_{b,g}$ true exactly when $b \in B_0$ and $f_0(b) = g$. Each formula of Y mentions only boys in B_0, and v satisfies it: formulas of type

[3] P. Hall, "On representatives of subsets", *Journal of the London Mathematical Society* 10 (1935), 26–30. Hall stated it in terms of *systems of distinct representatives*: choosing a different element from each of a family of finite sets. It is closely related to Kőnig's theorem on matchings in bipartite graphs.

[4] M. Hall Jr., "Distinct representatives of subsets", *Bulletin of the American Mathematical Society* 54 (1948), 922–926.

(a) because $f_0(b) \in N(b)$, of type (b) because f_0 is a function, and of type (c) because f_0 is injective. By Theorem 2.1.2, X is satisfiable.

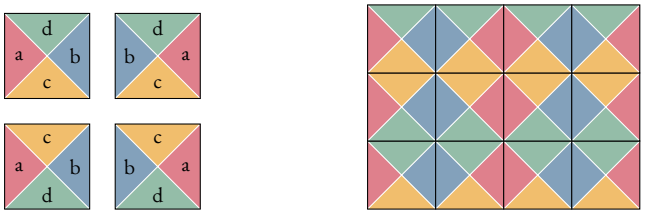
***Exercise** 3.2.4* Let $B = \{b_0, b_1, b_2, \dots\}$ and $\mathcal{G} = \{g_1, g_2, \dots\}$, where b_0 knows every girl and b_i knows only g_i for $i \geq 1$, as in Figure 3.2(b). Show that Hall's condition holds and no matching exists. Which step of the proof fails?

3.3 *Tiling the plane*

A *Wang tile* is a unit square with a colour on each of its four edges. Fix a finite set W of tiles, with copies of each tile available without limit and without rotation. For $w \in W$, write $\mathit{left}(w)$, $\mathit{right}(w)$, $\mathit{top}(w)$ and $\mathit{bottom}(w)$ for its edge colours. A *tiling* of a set $R \subseteq \mathbb{Z}^2$ of cells is a function $\tau : R \to W$ such that neighbouring edges match: whenever the cells involved lie in R,

$$\mathit{right}(\tau(x,y)) = \mathit{left}(\tau(x+1,y)), \qquad \mathit{top}(\tau(x,y)) = \mathit{bottom}(\tau(x,y+1)).$$

We say W *tiles* R if some tiling of R exists. Hao Wang introduced these tiles in 1961 and asked for an algorithm that decides, given W, if W tiles the plane.[5] Robert Berger showed in 1966 that no algorithm decides this.[6]



the tile set W part of a tiling of $\mathbb{Z}^2$

Figure 3.3 A set W of four tiles with edge colours a, b, c, d, and part of a tiling of the plane by W. Neighbouring edges have the same colour; the pattern repeats in both directions.

***Exercise** 3.3.1* Name the tiles of Figure 3.3 by their colours (left, right, bottom, top), and take the bottom-left cell of the patch to be $(0,0)$. Give $\tau(x,y)$ in terms of whether x and y are even or odd.

Theorem 3.3.2 *If W tiles the square $\{-n, \dots, n\}^2$ for every n, then W tiles $\mathbb{Z}^2$.*

Proof. Take an atomic proposition $t_{x,y,w}$ for each cell $(x,y) \in \mathbb{Z}^2$ and tile $w \in W$, read as "cell (x,y) holds tile w"; there are countably many, since $\mathbb{Z}^2 \times W$ is countable. Let X consist of the following formulas, for all x, y.

[5] H. Wang, "Proving theorems by pattern recognition II", *Bell System Technical Journal* 40 (1961), 1–41. Wang showed that such an algorithm would exist if every tile set that tiles the plane also tiles it periodically.

[6] R. Berger, "The undecidability of the domino problem", *Memoirs of the American Mathematical Society* 66 (1966). Berger's proof produced a tile set that tiles the plane but never periodically; it had 20,426 tiles. E. Jeandel and M. Rao found an aperiodic set of 11 Wang tiles in 2015, and showed that no smaller set exists.

(a) The disjunction of $t_{x,y,w}$ over $w \in W$.

(b) $\neg(t_{x,y,w} \wedge t_{x,y,w'})$ for distinct $w, w' \in W$.

(c) $\neg(t_{x,y,w} \wedge t_{x+1,y,w'})$ whenever $\mathit{right}(w) \neq \mathit{left}(w')$.

(d) $\neg(t_{x,y,w} \wedge t_{x,y+1,w'})$ whenever $\mathit{top}(w) \neq \mathit{bottom}(w')$.

A valuation satisfying X defines a tiling of $\mathbb{Z}^2$: let $\tau(x, y)$ be the unique w with $t_{x,y,w}$ true, which exists by (a) and (b); edges match by (c) and (d). By Theorem 2.1.2 it is enough to show that every $Y \subseteq_{\text{fin}} X$ is satisfiable. Such a Y mentions only cells in some square $\{-n, \dots, n\}^2$. Let τ_n be a tiling of that square, and make $t_{x,y,w}$ true exactly when (x, y) lies in the square and $\tau_n(x, y) = w$. Each formula of type (a) or (b) in Y holds because τ_n puts exactly one tile in each cell of the square. Each formula of type (c) or (d) in Y involves two neighbouring cells of the square, where τ_n matches edges. So this valuation satisfies Y.

***Exercise** 3.3.3* Show that W tiles $\mathbb{Z}^2$ if and only if it tiles the quadrant $\{0, 1, 2, \dots\}^2$. (Hint: tilings can be translated, and the quadrant contains a translate of every square.)

Further exercises

***Exercise** 3.3.4* A *partial order* $\leq$ on a set A is reflexive, antisymmetric and transitive; it is *linear* if any two elements are comparable. Every partial order on a finite set extends to a linear order. Use compactness to show that every partial order on a countable set extends to a linear order. (Use atoms $p_{a,b}$ for $a, b \in A$, read as "a comes before b in the new order".)

First-Order Logic

"When I *use a word," Humpty Dumpty said, in rather a scornful tone,*
"it means just what I choose it to mean, neither more nor less."
Lewis Carroll, *Through the Looking-Glass*

In this chapter, we introduce first-order logic and prove its compactness theorem. The proofs of Chapter 3 needed separate atomic propositions for each vertex or cell of one infinite object. A first-order sentence uses variables and quantifiers, so a single sentence can state a property of every graph, or of every group. A structure, such as a graph or a group, is a set of elements with relations, functions and constants on it. First-order logic adds symbols for these relations, functions and constants to the propositional connectives, together with a quantifier $\exists$ ("there exists") over elements. Quantifiers first appear in Gottlob Frege's *Begriffsschrift* of 1879.[1] After defining the syntax and semantics, we prove compactness by Henkin's construction, which follows the proof of Chapter 2 step by step. This chapter follows Chapter 4 of Mukund and Suresh's notes.

An example Take a binary relation symbol E for the edges of a graph. Using notation defined below, the sentence

$$\forall x \, \forall y \, (E(x, y) \supset E(y, x))$$

[1] G. Frege, *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, Louis Nebert, Halle, 1879. Frege's notation was two-dimensional and is no longer used; the quantifier symbols $\forall$ and $\exists$ came later: $\forall$ from Gentzen (1935) and $\exists$ from Peano (1897).

says that edges are symmetric, and the sentence $\forall x\, \exists y\, \exists z\, (E(x,y) \wedge E(x,z) \wedge \neg(y \approx z))$ says that every vertex has at least two neighbours. A graph *satisfies* a sentence if the sentence is true when E is read as its edge relation.

4.1 Syntax

Definition 4.1.1 (Languages) A *first-order language* is a triple $L = (R, F, C)$ of sets of *relation symbols*, *function symbols* and *constant symbols*, together with an *arity* $n \geq 1$ for each relation and function symbol, the number of arguments it takes. We also fix a countably infinite set Var of *variables*, written x, y, z. The symbols of L are those in R, F, C and Var, the equality symbol $\approx$, the connectives $\neg$ and $\vee$, the quantifier $\exists$, the comma and the two parentheses; all are distinct.

Until Section 5.1 we assume that R, F and C are countable. Binary symbols such as $<$ and $+$ may be written between their arguments. For example, the language of graphs has one binary relation symbol E and no function or constant symbols, and the language of groups has a binary function symbol $\cdot$ and a constant symbol e.

Definition 4.1.2 (Terms) The *terms* over L form the smallest set of strings containing every constant and every variable, and containing $f(t_1, \dots, t_n)$ whenever $f \in F$ has arity n and $t_1, \dots, t_n$ are terms. A term with no variables is *closed*.

In the language of groups, $x \cdot e$ and $(x \cdot y) \cdot z$ are terms, and $e \cdot e$ is a closed term.

Definition 4.1.3 (Formulas) An *atomic formula* is a string $r(t_1, \dots, t_n)$, with $r \in R$ of arity n and terms $t_1, \dots, t_n$, or $t_1 \approx t_2$, with terms t_1, t_2. The set Φ_L of *formulas* is the smallest set of strings that contains the atomic formulas, contains $(\neg\varphi)$ and $(\varphi \vee \psi)$ whenever it contains φ and ψ, and contains $\exists x\, \varphi$ whenever it contains φ and $x \in Var$.

We write φ, ψ, χ for first-order formulas and keep α, β, γ for propositional ones. Equality is written $\approx$, since $\equiv$ already means *if and only if*; Mukund and Suresh write $\equiv$. As in Chapter 1, every term and every formula is built in exactly one way, so we may define functions on them by recursion and prove facts about them by structural induction. We drop parentheses by the same conventions as there. The connectives $\wedge$, $\supset$, $\equiv$ are abbreviations as before, and $\forall x\, \varphi \stackrel{\text{def}}{=} \neg\exists x\, \neg\varphi$.

Definition 4.1.4 (Free variables) For a term t, let $Var(t)$ be the set of variables occurring in t. The set $FV(\varphi)$ of *free variables* of a formula is defined by recursion:

$$FV(r(t_1, \dots, t_n)) = Var(t_1) \cup \cdots \cup Var(t_n), \qquad FV(t_1 \approx t_2) = Var(t_1) \cup Var(t_2),$$
$$FV(\neg\varphi) = FV(\varphi), \qquad FV(\varphi \vee \psi) = FV(\varphi) \cup FV(\psi), \qquad FV(\exists x\, \varphi) = FV(\varphi) \setminus \{x\}.$$

A *sentence* is a formula φ with $FV(\varphi) = \emptyset$. We write $\varphi(x)$ for a formula with $FV(\varphi) \subseteq \{x\}$.

For example, $FV(E(x,y) \vee \exists x\, E(x,x)) = \{x, y\}$: the x in $E(x,y)$ is free, and the two occurrences of x in $E(x,x)$ are bound by $\exists x$.

Definition 4.1.5 (Substitution) Let x be a variable and t a closed term. For a term u, the term $u[t/x]$ replaces each occurrence of x in u by t. For a formula χ, the formula $\chi[t/x]$ is defined by recursion:

$$r(t_1,\dots,t_n)[t/x] = r(t_1[t/x],\dots,t_n[t/x]), \qquad (t_1 \approx t_2)[t/x] = (t_1[t/x] \approx t_2[t/x]),$$
$$(\neg\varphi)[t/x] = \neg(\varphi[t/x]), \qquad (\varphi \vee \psi)[t/x] = \varphi[t/x] \vee \psi[t/x],$$
$$(\exists x\,\varphi)[t/x] = \exists x\,\varphi, \qquad (\exists y\,\varphi)[t/x] = \exists y\,(\varphi[t/x]) \text{ for } y \neq x.$$

We write $\varphi(t)$ for $\varphi(x)[t/x]$.

So $\chi[t/x]$ replaces the free occurrences of x in χ by t. Since t is closed, no variable of t can be captured by a quantifier. By structural induction, $FV(\chi[t/x]) = FV(\chi) \setminus \{x\}$, so in particular $\varphi(t)$ is a sentence.

***Exercise** 4.1.6* Find the free variables of $\exists x\,(E(x,y) \wedge \exists y\,E(y,z))$. Is it a sentence? Compute $(E(x,y) \vee \exists x\,E(x,x))[c/x]$ for a constant c.

4.2 Semantics

Definition 4.2.1 (Structures) A *structure* for L is a pair $M = (D, \iota)$, where D is a nonempty set, the *domain* of M, and ι assigns a relation $r^M = \iota(r) \subseteq D^n$ to each $r \in R$ of arity n, a function $f^M = \iota(f) : D^n \to D$ to each $f \in F$ of arity n, and an element $c^M = \iota(c) \in D$ to each $c \in C$.

Example 4.2.2 Let L have one binary relation symbol E. The structure M with $D = \{1,2,3\}$ and $E^M = \{(1,2),(2,1),(2,3),(3,2)\}$ is the path in Figure 4.1. It satisfies the symmetry sentence of the opening example. It does not satisfy the second sentence there, since vertex 1 has only one neighbour.

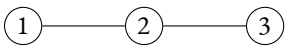


Figure 4.1 The structure of Example 4.2.2: each line between i and j stands for the two pairs (i,j) and (j,i) in E^M.

To evaluate a formula with free variables we must also say what the variables stand for.

Definition 4.2.3 (Assignments and values) An *assignment* in M is a function $\sigma : Var \to D$. For $x \in Var$ and $s \in D$, the assignment $\sigma[x \mapsto s]$ sends x to s and agrees with σ elsewhere. The *value* $t^{M,\sigma} \in D$ of a term t is defined by recursion:

$$c^{M,\sigma} = c^M, \qquad x^{M,\sigma} = \sigma(x), \qquad f(t_1,\dots,t_n)^{M,\sigma} = f^M(t_1^{M,\sigma},\dots,t_n^{M,\sigma}).$$

By structural induction, the value of a closed term t does not depend on σ; we write it t^M.

Definition 4.2.4 (Satisfaction) The relation $(M,\sigma) \vDash \varphi$, read "$\varphi$ is true in M under σ", is defined by recursion on φ.

- $(M,\sigma) \models t_1 \approx t_2$ if $t_1^{M,\sigma} = t_2^{M,\sigma}$.
- $(M,\sigma) \models r(t_1,\dots,t_n)$ if $(t_1^{M,\sigma},\dots,t_n^{M,\sigma}) \in r^M$.
- $(M,\sigma) \models \neg\varphi$ if $(M,\sigma) \not\models \varphi$.
- $(M,\sigma) \models \varphi \vee \psi$ if $(M,\sigma) \models \varphi$ or $(M,\sigma) \models \psi$.
- $(M,\sigma) \models \exists x\,\varphi$ if $(M,\sigma[x \mapsto s]) \models \varphi$ for some $s \in D$.

For example, in the path M of Example 4.2.2, let $\sigma(x) = 1$. Then $(M,\sigma) \models \exists y\, E(x,y)$, with $s = 2$, since $(1,2) \in E^M$. But $(M,\sigma) \not\models \exists y\,\exists z\,(E(x,y) \wedge E(x,z) \wedge \neg(y \approx z))$, because 2 is the only s with $(1,s) \in E^M$.

***Exercise** 4.2.5* Does the path M of Example 4.2.2 satisfy $\exists x\,\forall y\,(\neg(x \approx y) \supset E(x,y))$? Does it satisfy $\forall x\,\exists y\,E(x,y)$?

Lemma 4.2.6 *Let M be a structure for L.*

(1) If assignments σ and σ' agree on $FV(\varphi)$, then $(M,\sigma) \models \varphi$ if and only if $(M,\sigma') \models \varphi$.

(2) Substitution lemma. *For every formula χ, variable x, assignment σ and closed term t,*

$$(M,\sigma) \models \chi[t/x] \quad \textit{if and only if} \quad (M,\sigma[x \mapsto t^M]) \models \chi.$$

(3) Let M' be a structure for a larger language that has the same domain as M and agrees with M on the symbols of L. Then M and M' satisfy the same formulas of L under any assignment.

Proof. Each part is proved by structural induction on the formula; see Exercise 4.2.9. Part (1) is the first-order form of Proposition 1.3.3.

By (1), whether a sentence φ is true in M does not depend on the assignment, and we write $M \models \varphi$. By (2), $M \models \varphi(t)$ if and only if $(M,\sigma[x \mapsto t^M]) \models \varphi(x)$ for any assignment σ.

Definition 4.2.7 (Models and consequence) Let X be a set of sentences. A structure M *satisfies* X, written $M \models X$, if $M \models \varphi$ for every $\varphi \in X$; then M is a *model* of X. The set X is *satisfiable* if it has a model, and, as in Section 2.1, *finitely satisfiable* if every $Y \subseteq_{\text{fin}} X$ is satisfiable. For a sentence φ, we write $X \models \varphi$ if every model of X satisfies φ.

Some sentences speak only about the size of the domain. For $n \geq 1$, let

$$\lambda_n = \exists x_1 \cdots \exists x_n \bigwedge_{i<j} \neg(x_i \approx x_j)$$

(for $n = 1$, read the empty conjunction as $x_1 \approx x_1$). The sentence λ_n holds in M if and only if D has at least n elements.

***Exercise** 4.2.8* Show that $(M, \sigma) \vDash \forall x\, \varphi$ if and only if $(M, \sigma[x \mapsto s]) \vDash \varphi$ for every $s \in D$. Check that λ_n holds in M if and only if D has at least n elements.

***Exercise** 4.2.9* Prove Lemma 4.2.6 by structural induction, for all assignments at once. (In part (2), for $\exists y\, \chi$, treat $y = x$ and $y \neq x$ separately.)

Compactness The compactness theorem of Chapter 2 has a first-order version. In 1928 David Hilbert and Wilhelm Ackermann asked whether every first-order sentence that is true in all structures can be proved.[2] Gödel's doctoral thesis answered yes, and compactness follows from his result because a proof uses only finitely many hypotheses, as Section 2.1 describes.

Theorem 4.2.10 (Compactness; Gödel, 1930, and Mal'cev, 1936) *A set X of sentences of L is satisfiable if and only if it is finitely satisfiable.*

If $M \vDash X$ and $Y \subseteq X$, then $M \vDash Y$, so every satisfiable set is finitely satisfiable. For the converse, as in Chapter 2, we extend a finitely satisfiable set as far as possible and read a model off the result.

4.3 Henkin's construction

This section parallels Section 2.2. We build a model from a set of sentences alone, so its elements must come from the syntax, and we take them to be closed terms. The language therefore first needs enough constants. Suppose L has one constant c and one unary relation symbol r. The set $\{\exists x\, r(x),\ \neg r(c)\}$ is satisfiable, but the only closed term is c, so a model made of closed terms has a single element c, which is not in r. A new constant naming an element of r removes the problem. The construction is due to Leon Henkin, who used it in 1949 to give a new proof of Gödel's completeness theorem.[3] We use it to prove compactness without any proof system.

The witnessing expansion Let $L_0 = L$. Given L_n, form L_{n+1} by adding a new constant c_φ for each formula $\varphi(x)$ of L_n that does not yet have one; c_φ is meant to name an element satisfying $\varphi(x)$, if one exists. A formula that mentions a new constant needs a constant of its own, so the step is repeated. Let L_H be the union of the L_n. Here $\varphi(x)$ counts as the pair of the formula and the variable x, so a sentence is $\varphi(x)$ for every variable x and gets one constant for each; the notation c_φ leaves x out. Every formula of L_H uses finitely many symbols, so it lies in some L_n, and every formula $\varphi(x)$ of L_H has its constant c_φ. By induction on n, every L_n has countably many symbols: if L_n does, then it has countably many formulas, so L_{n+1} adds countably many constants. Hence L_H, a countable union of countable sets, also has countably many symbols. The *Henkin axioms* are the sentences

$$\exists x\, \varphi(x) \supset \varphi(c_\varphi)$$

[2] D. Hilbert and W. Ackermann, *Grundzüge der theoretischen Logik*, Springer, Berlin, 1928.

[3] L. Henkin, "The completeness of the first-order functional calculus", *Journal of Symbolic Logic* 14 (1949), 159–166. The paper came from Henkin's 1947 doctoral thesis at Princeton, written under Alonzo Church. Henkin's method is now the standard proof of completeness, and is used in many other logics.

for all formulas $\varphi(x)$ of L_H; let H be the set of them.

***Exercise** 4.3.1* Let L have one unary relation symbol r and no constants. Name two constants of L_1, and one constant of L_2 that is not in L_1.

Lemma 4.3.2 *If X is a finitely satisfiable set of sentences of L, then the set $X \cup H$ is also finitely satisfiable.*

Proof. Let Z be a finite subset of $X \cup H$. The finite set $Z \cap X$ has a model M_0, which we extend to each L_n in turn. Given a structure M_n for L_n, let M_{n+1} be the structure for L_{n+1} with the same domain that agrees with M_n on L_n. It sends each new constant c_φ to some s with $(M_n, \sigma[x \mapsto s]) \vDash \varphi(x)$ if there is one, and to any element otherwise. By Lemma 4.2.6(1), whether such an s exists does not depend on the assignment σ. Each M_{n+1} agrees with M_n on L_n, so there is a structure M_H for L_H that agrees with M_n on the symbols of L_n for every n. By Lemma 4.2.6(3), M_H satisfies $Z \cap X$.

It remains to check that M_H satisfies every Henkin axiom. Take a Henkin axiom whose constant c_φ was added to L_{n+1}, so that $\varphi(x)$ is a formula of L_n, and suppose $M_H \vDash \exists x\, \varphi(x)$. Then $M_n \vDash \exists x\, \varphi(x)$ by Lemma 4.2.6(3). So some s satisfies $\varphi(x)$ in M_n, and c_φ was sent to such an s. Applying Lemma 4.2.6(3) again gives $(M_H, \sigma[x \mapsto c_\varphi^{M_H}]) \vDash \varphi(x)$, and the substitution lemma then gives $M_H \vDash \varphi(c_\varphi)$. So M_H satisfies every Henkin axiom, and $M_H \vDash Z$.

Maximal sets As in Section 2.2, a finitely satisfiable set Σ of sentences of L_H is *maximal* if, for every sentence $\varphi \notin \Sigma$, the set $\Sigma \cup \{\varphi\}$ is not finitely satisfiable.

Example 4.3.3 As in Example 2.2.2, for a structure M for L_H the set $\Sigma_M = \{\varphi : M \vDash \varphi\}$ of sentences of L_H is maximal finitely satisfiable. If M interprets the constants c_φ as in the proof of Lemma 4.3.2, then $H \subseteq \Sigma_M$. The proof of compactness goes the other way, and builds a structure from a maximal finitely satisfiable set.

Lemma 4.3.4 (Lindenbaum) *Every finitely satisfiable set of sentences of L_H is contained in a maximal finitely satisfiable set.*

Proof. The proofs of Lemmas 2.2.3 and 2.2.4 carry over word for word, with sentences of L_H in place of formulas and structures in place of valuations. They need a list $\varphi_1, \varphi_2, \ldots$ of all sentences of L_H, which exists because L_H has countably many symbols.

Lemma 4.3.5 *Let Σ be a maximal finitely satisfiable set of sentences of L_H with $H \subseteq \Sigma$. For all sentences φ, ψ of L_H:*

(a) exactly one of φ and $\neg\varphi$ is in Σ;

(b) $\varphi \vee \psi \in \Sigma$ if and only if $\varphi \in \Sigma$ or $\psi \in \Sigma$;

(c) if $Y \vDash \psi$ for some finite $Y \subseteq \Sigma$, then $\psi \in \Sigma$;

(d) $\exists x\, \varphi(x) \in \Sigma$ if and only if $\varphi(t) \in \Sigma$ for some closed term t.

Proof. (a) and (b) are proved as in Lemma 2.2.5. (c) Otherwise $\neg\psi \in \Sigma$ by (a), and the finite set $Y \cup \{\neg\psi\} \subseteq \Sigma$ is not satisfiable, since a model of Y satisfies ψ. This contradicts the finite satisfiability of Σ. (d) If $\exists x\, \varphi(x) \in \Sigma$, then Σ also contains the Henkin axiom $\exists x\, \varphi(x) \supset \varphi(c_\varphi)$, and these two sentences entail $\varphi(c_\varphi)$, so $\varphi(c_\varphi) \in \Sigma$ by (c). Conversely, $\{\varphi(t)\} \vDash \exists x\, \varphi(x)$: if a structure M satisfies $\varphi(t)$, then $(M, \sigma[x \mapsto t^M]) \vDash \varphi(x)$ by the substitution lemma. So if $\varphi(t) \in \Sigma$, then $\exists x\, \varphi(x) \in \Sigma$ by (c).

The term model Let Σ be as in Lemma 4.3.5. For closed terms t, u of L_H, let $t \sim u$ if $t \approx u \in \Sigma$. By Lemma 4.3.5(c), $\sim$ is an equivalence relation: for instance, $\{t \approx u,\ u \approx w\} \vDash t \approx w$, so $t \sim u$ and $u \sim w$ give $t \sim w$. Write $[t]$ for the class of t, and define a structure M for L_H:

- D is the set of classes $[t]$ of closed terms of L_H. It is nonempty, since L_H contains the constant $c_{x \approx x}$.
- $c^M = [c]$ for each constant c, and $f^M([t_1], \dots, [t_n]) = [f(t_1, \dots, t_n)]$.
- $([t_1], \dots, [t_n]) \in r^M$ if and only if $r(t_1, \dots, t_n) \in \Sigma$.

By Lemma 4.3.5(c) again, f^M and r^M do not depend on the chosen representatives. By induction on t, we have $t^M = [t]$ for every closed term t.

***Exercise** 4.3.6* Check that $\sim$ is an equivalence relation, that f^M and r^M are well defined, and that $t^M = [t]$ for closed terms t.

Lemma 4.3.7 (Truth lemma) *For every sentence φ of L_H, $M \vDash \varphi$ if and only if $\varphi \in \Sigma$.*

Proof. Structural induction, used for Lemma 2.2.6, does not suffice here, since $\psi(t)$ below is not a subformula of $\exists x\, \psi(x)$. We use induction on the number of occurrences of $\neg$, $\vee$ and $\exists$ in φ. For atomic sentences this is the definition of M. For instance, since $t^M = [t]$ and $u^M = [u]$, we have $M \vDash t \approx u$ if and only if $[t] = [u]$, which holds if and only if $t \approx u \in \Sigma$. The cases $\neg$ and $\vee$ follow from the induction hypothesis and Lemma 4.3.5(a) and (b). Let $\varphi = \exists x\, \psi(x)$. Every element of D is t^M for a closed term t, so by the substitution lemma, $M \vDash \varphi$ if and only if $M \vDash \psi(t)$ for some closed term t. Each $\psi(t)$ has fewer occurrences than φ, so by the induction hypothesis $M \vDash \psi(t)$ if and only if $\psi(t) \in \Sigma$. By Lemma 4.3.5(d), $\psi(t) \in \Sigma$ for some t if and only if $\varphi \in \Sigma$.

4.4 Proof of the compactness theorem

Proof of Theorem 4.2.10. The proof parallels the proof of Theorem 2.1.2 in Section 2.3, with Lemmas 4.3.4 and 4.3.7 in place of Lemmas 2.2.4 and 2.2.6; the only new step is adding the Henkin axioms. Let X be finitely satisfiable. By Lemma 4.3.2, $X \cup H$ is finitely satisfiable. By Lemma 4.3.4, $X \cup H$ is contained in a maximal finitely satisfiable set Σ. By Lemma 4.3.7, the term model M of Σ satisfies Σ, and so X. Forgetting the new constants gives a structure for L that satisfies X, by Lemma 4.2.6(3).

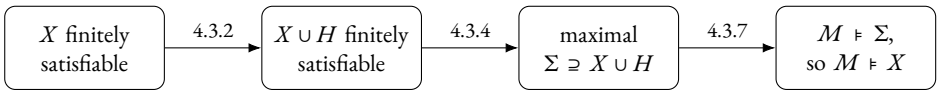


Figure 4.2 The steps in the proof of Theorem 4.2.10, labelled by lemma as in Figure 2.2; M is the term model of Σ.

As in Section 2.4, compactness has a consequence form.

Lemma 4.4.1 *For every set Z of sentences and every sentence φ, $Z \models \varphi$ if and only if $Z \cup \{\neg\varphi\}$ is not satisfiable.*

Proof. A structure M satisfies $Z \cup \{\neg\varphi\}$ exactly when it satisfies Z and $M \not\models \varphi$. Such a structure exists if and only if $Z \not\models \varphi$.

Theorem 4.4.2 *Let X be a set of sentences and φ a sentence. Then $X \models \varphi$ if and only if $Y \models \varphi$ for some $Y \subseteq_{\text{fin}} X$.*

Proof. ($\Leftarrow$) Let $Y \subseteq_{\text{fin}} X$ with $Y \models \varphi$. If $M \models X$, then $M \models Y$, so $M \models \varphi$.

($\Rightarrow$) Suppose $X \models \varphi$. By Lemma 4.4.1, $X \cup \{\neg\varphi\}$ is not satisfiable. By Theorem 4.2.10, some $Z \subseteq_{\text{fin}} X \cup \{\neg\varphi\}$ is not satisfiable. Let $Y = Z \setminus \{\neg\varphi\}$, a finite subset of X. Then $Y \cup \{\neg\varphi\} \supseteq Z$ is not satisfiable either, so $Y \models \varphi$ by Lemma 4.4.1.

The term model in the proof of Theorem 4.2.10 has countably many elements, since L_H has countably many closed terms. So a satisfiable set of sentences of a countable language has a countable model. This is the (downward) *Löwenheim–Skolem theorem*, proved by Leopold Löwenheim in 1915 and Thoralf Skolem in 1920.[4] If the axioms of set theory have a model, the theorem gives them a countable model, even though the axioms prove that uncountable sets exist. This is *Skolem's paradox*. There is no contradiction. A set in the model can be countable from outside and still be uncountable inside the model, because the model contains no bijection between that set and the natural numbers.

***Exercise** 4.4.3* Let X be a set of sentences with models of every finite size $n \geq 1$. Show that X has an infinite model. (Consider $X \cup \{\lambda_n : n \geq 1\}$.) Conclude that no set of sentences is satisfied by exactly the finite structures.

[4] L. Löwenheim, "Über Möglichkeiten im Relativkalkül", *Mathematische Annalen* 76 (1915), 447–470; T. Skolem, "Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen", *Videnskapsselskapets Skrifter, I. Matematisk-naturvidenskabelig Klasse* 4 (1920).

Applications II

Complete disorder is impossible.
Theodore Motzkin

In this chapter, we apply first-order compactness to three problems: finding models of arbitrarily large size, deducing Ramsey's theorem for finite graphs from the version for infinite graphs, and carrying statements about fields of characteristic 0 over to fields of large prime characteristic. Most of the proofs add sentences to a set and check that the larger set is finitely satisfiable. To force a model to be large, we add new constants with sentences saying that they are distinct; to force a model to be infinite, we add the sentences λ_n of Section 4.2. The last application uses the consequence form of compactness, Theorem 4.4.2.

5.1 Large models

Section 4.4 gives countable models. To get uncountable models from compactness, we add uncountably many new constants.

In the proof of Theorem 4.2.10, countability is used only in Lemma 4.3.4, to list the sentences of L_H. With uncountably many new constants the language is uncountable, and its sentences cannot be listed in this way. Zorn's lemma gives a maximal finitely satisfiable set without listing the sentences. It states that if every chain (a family of sets linearly ordered by $\subseteq$) in a nonempty family of sets has an upper bound in the family, then the family has a maximal member. We apply it to the family of finitely satisfiable

sets containing $X \cup H$. The empty chain has the upper bound $X \cup H$, and any other chain has its union as an upper bound. The union is finitely satisfiable, because each sentence of a finite subset lies in some member of the chain, and the largest of these finitely many members contains them all. Zorn's lemma is equivalent to the axiom of choice and is beyond the course.[1] With this change, the proof of Theorem 4.2.10 gives first-order compactness for languages of any size.

Theorem 5.1.1 (Upward Löwenheim–Skolem; Tarski, 1928) *Let X be a set of sentences with an infinite model, and let I be any set. Then X has a model with an injective map from I into its domain.*

Proof. Add a new constant c_i for each $i \in I$, and let $X^+ = X \cup \{\neg(c_i \approx c_j) : i, j \in I,\ i \neq j\}$. A finite subset of X^+ mentions finitely many new constants. Interpret them as distinct elements of an infinite model of X, and the other c_i arbitrarily. The sentences of X in the subset still hold by Lemma 4.2.6(3). The inequalities in the subset hold because they mention only constants sent to distinct elements. By compactness for languages of any size, X^+ has a model M. Forgetting the constants c_i gives a model of X with the same elements, by Lemma 4.2.6(3). The map $i \mapsto c_i^M$ is injective because $M \models \neg(c_i \approx c_j)$ for $i \neq j$.

Taking $I = \mathbb{R}$ shows that every set of sentences with an infinite model has an uncountable model. Now let M be a model of X and take I to be the power set of the domain of M. By Cantor's theorem the power set of a set has no injection into the set, so this gives a model of X larger than M. Hence no set of first-order sentences with an infinite model has all its models of one size.[2]

***Exercise** 5.1.2* Let Z be the set of sentences true in the countably infinite graph with no edges. Show that every model of Z is an infinite graph with no edges, and that Z has a model with uncountably many elements.

5.2 Ramsey's theorem

Let $G = (V, E)$ be a graph. A set $H \subseteq V$ is a *clique* if any two distinct vertices of H are joined by an edge, and *independent* if no two are. It is *homogeneous* if it is a clique or independent. Ramsey's theorem says that every large graph has a large homogeneous set, whatever its edges. Frank Ramsey proved it in 1930, as a step in a paper on the decision

[1] M. Zorn, "A remark on method in transfinite algebra", *Bulletin of the American Mathematical Society* 41 (1935), 667–670. Kuratowski had proved an equivalent principle in 1922.

[2] Tarski proved the upward theorem in a seminar in 1928. The first printed proof is in Mal'cev's 1936 paper, cited in Section 2.1, which credits Tarski. A. Tarski and R. L. Vaught, "Arithmetical extensions of relational systems", *Compositio Mathematica* 13 (1957), 81–102, gave the version with elementary extensions. The theorem is sometimes called the Löwenheim–Skolem–Tarski theorem. Together with a form of the downward theorem for uncountable languages, it shows that a set of sentences of a countable language with an infinite model has models of every infinite size.

problem for a fragment of first-order logic; he died that year, aged 26.[3] Paul Erdős and George Szekeres rediscovered it in 1935 and made it widely known.[4]

Theorem 5.2.1 (Ramsey, 1930) *Every graph with infinitely many vertices has an infinite homogeneous set.*

Proof. Let $V_0 = V$ and choose $u_0 \in V_0$. Suppose V_i is infinite and $u_i \in V_i$. The infinite set $V_i \setminus \{u_i\}$ splits into the neighbours of u_i and the non-neighbours, and one of the two parts is infinite. Let V_{i+1} be an infinite part, colour u_i *red* if V_{i+1} consists of neighbours of u_i and *blue* otherwise, and choose $u_{i+1} \in V_{i+1}$.

For $i < j$ we have $u_j \in V_j \subseteq V_{i+1}$, so $u_j \neq u_i$. The set V_{i+1} consists entirely of neighbours or entirely of non-neighbours of u_i, so u_i is joined to u_j exactly when u_i is red. There are infinitely many u_i and only two colours, so infinitely many have the same colour. If infinitely many are red, they form a clique, since each red u_i is joined to every later u_j. If infinitely many are blue, they form an independent set in the same way.

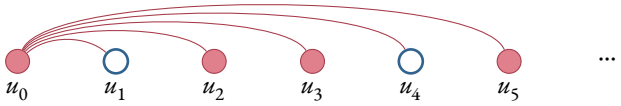


Figure 5.1 The sequence in the proof of Theorem 5.2.1. A red vertex (filled) is joined to every later vertex; a blue vertex (open) to none. Only the edges at u_0 are drawn.

***Exercise** 5.2.2* Let $V = \{0, 1, 2, \dots\}$, with distinct m and n joined when $m + n$ is even. Run the proof of Theorem 5.2.1 with $u_0 = 0$, always choosing the neighbours when they form an infinite set and taking the least element of V_{i+1} as u_{i+1}. What homogeneous set results?

The finite version The finite version states that every sufficiently large graph has a homogeneous set of a given size. It follows from the infinite version by compactness. Let L have one binary relation symbol E, and let Γ consist of the two sentences

$$\forall x \neg E(x, x) \quad \text{and} \quad \forall x \forall y \, (E(x, y) \supset E(y, x)),$$

whose models are the graphs.

Theorem 5.2.3 (Ramsey, finite version) *For every $k \geq 1$ there is an n such that every graph with n vertices has a homogeneous set with k elements.*

[3] F. P. Ramsey, "On a problem of formal logic", *Proceedings of the London Mathematical Society* (2) 30 (1930), 264–286. Ramsey also made lasting contributions to economics and the philosophy of probability.

[4] P. Erdős and G. Szekeres, "A combinatorial problem in geometry", *Compositio Mathematica* 2 (1935), 463–470. They used it to show that among enough points in the plane, no three on a line, some k form a convex polygon. Esther Klein had shown that any five such points include four in convex position, and asked for the general statement; she later married Szekeres, and the problem is known as the *happy ending problem*.

Proof. Let η_k say that there is no homogeneous set with k elements:

$$\neg\exists x_1 \cdots \exists x_k \left(\theta \wedge \bigwedge_{i<j} E(x_i, x_j)\right)$$
$$\wedge\, \neg\exists x_1 \cdots \exists x_k \left(\theta \wedge \bigwedge_{i<j} \neg E(x_i, x_j)\right),$$

where θ is $\bigwedge_{i<j} \neg(x_i \approx x_j)$ and the big conjunctions are finite.

Suppose the theorem fails for some k, so that for every n there is a graph with n vertices and no homogeneous set with k elements. Then $k \geq 2$, since $n = 1$ works for $k = 1$. Let $X = \Gamma \cup \{\eta_k\} \cup \{\lambda_n : n \geq 1\}$. A finite subset of X contains λ_n only for $n \leq N$, for some N. Take a graph G with N vertices and no homogeneous set with k elements. It satisfies Γ because it is a graph, η_k by the choice of G, and λ_n for every $n \leq N$ because it has N vertices, so it satisfies the finite subset. By Theorem 4.2.10, X has a model. This model satisfies Γ and every λ_n, so it is a graph with infinitely many vertices. It also satisfies η_k, so it has no homogeneous set with k elements and hence no infinite homogeneous set. This contradicts Theorem 5.2.1.

The proof gives no bound on n. The least n that works for k is the *Ramsey number* $R(k)$; exact values are known only for $k \leq 4$, with $R(4) = 18$.[5]

***Exercise** 5.2.4* Show that $R(3) = 6$: every graph with 6 vertices has a homogeneous set with 3 elements, and the 5-cycle (vertices $1, \ldots, 5$, with i joined to $i+1$ and 5 joined to 1) does not. Deduce that no $n \leq 5$ works, by deleting vertices from the 5-cycle.

***Exercise** 5.2.5* Colour each pair of distinct elements of a set with one of m colours. State and prove an infinite version of Ramsey's theorem for m colours, and deduce the finite version by compactness. (Use one relation symbol for each colour, with sentences saying that each pair of distinct elements has exactly one colour.)

5.3 *Fields of large characteristic*

This section assumes familiarity with fields. Let L have constants 0, 1 and binary function symbols $+$, $\cdot$, written between their arguments. The field axioms are finitely many sentences of L; let Σ_F be their set. A field has *characteristic* p, for a prime p, if $1 + \cdots + 1 = 0$ with p ones; at most one prime has this property. A field has characteristic 0 if no such sum is 0, and every field has characteristic 0 or a prime. Let χ_p be the sentence $1 + \cdots + 1 \approx 0$ with p ones. The fields of characteristic 0 are the models of $\Sigma_F \cup \{\neg\chi_p : p \text{ prime}\}$.

Theorem 5.3.1 *If a sentence φ of L holds in every field of characteristic 0, then there is N such that φ holds in every field of characteristic $p > N$.*

Proof. By assumption $\Sigma_F \cup \{\neg\chi_p : p \text{ prime}\} \models \varphi$. By Theorem 4.4.2, some finite subset Y of this set has $Y \models \varphi$; it mentions $\neg\chi_p$ only for primes $p \leq N$, for some N. Let F be a field

[5] For $R(5)$ only bounds are known: $43 \leq R(5) \leq 46$, the upper bound proved by computer in 2024.

of characteristic $p > N$. It satisfies Σ_Γ. Since p is the only prime q for which χ_q holds in F, and $p > N$, the field F satisfies $\neg\chi_q$ for every prime $q \leq N$. So F satisfies Y, and hence φ.

For example, $\neg(1 + 1 \approx 0)$ holds in every field of characteristic 0 and fails only in characteristic 2, so $N = 2$ works.

***Exercise** 5.3.2* For the sentence $\neg(1 + 1 + 1 \approx 0)$, find the least N such that it holds in every field of characteristic $p > N$.

***Exercise** 5.3.3* Show that no set of sentences of L has as its models exactly the fields of positive characteristic. (If X were such a set, apply compactness to $X \cup \{\neg\chi_p : p \text{ prime}\}$.)

***Exercise** 5.3.4* This exercise assumes familiarity with groups. Let L have a constant e and a binary function symbol $\cdot$. A group is *torsion* if every element has finite order. Show that no set of sentences of L has as its models exactly the torsion groups. (If X were such a set, add a constant c to L, and add to X the sentences $\neg(c^n \approx e)$ for $n \geq 1$, where c^n is the product of n copies of c.)

Limits of First-Order Logic

The limits of my language mean the limits of my world.
Ludwig Wittgenstein, *Tractatus Logico-Philosophicus* 5.6

In this chapter, we use compactness to show that some properties cannot be expressed in first-order logic. To show that no set of sentences expresses a property, we suppose that one does, add sentences that the property forbids, and use compactness to show that the larger set still has a model. That model satisfies the original sentences but lacks the property, a contradiction. We apply this to connectivity of graphs, and then to the natural numbers, where compactness gives a model of every sentence true in $\mathbb{N}$ with an element larger than every natural number.

6.1 Connectivity is not first-order

Let L have one binary relation symbol E, and let Γ be the two sentences of Section 5.2 whose models are the graphs. A *walk of length n* from s to s' in a graph is a sequence $s = s_0, s_1, \ldots, s_n = s'$ in which consecutive vertices are joined. A graph is *connected* if any two vertices are joined by a walk.

Theorem 6.1.1 *There is no set X of sentences of L such that a graph satisfies X if and only if it is connected.*

Proof. Suppose X were such a set. Add two constants a and b to L, and for each $n \geq 0$ let δ_n say that there is no walk of length n from a to b:

$$\delta_n = \neg\exists x_0 \cdots \exists x_n \left(x_0 \approx a \wedge x_n \approx b \wedge E(x_0, x_1) \wedge \ldots \wedge E(x_{n-1}, x_n)\right).$$

Let $X^+ = \Gamma \cup X \cup \{\delta_n : n \geq 0\}$.

A finite subset of X^+ mentions δ_n only for $n \leq N$, for some N. Consider the path with vertices $0, 1, \ldots, N+1$, edges between consecutive numbers, $a = 0$ and $b = N + 1$. The path is a connected graph, so it satisfies Γ and X. Each step of a walk moves between consecutive numbers, so every walk from a to b has length at least $N + 1$. Hence the path satisfies $\delta_0, \ldots, \delta_N$ (Figure 6.1), and therefore the finite subset. By Theorem 4.2.10, X^+ has a model M. The structure obtained from M by forgetting a and b satisfies Γ and X by Lemma 4.2.6(3), so it is a connected graph. But M satisfies every δ_n, so there is no walk from a^M to b^M, a contradiction.

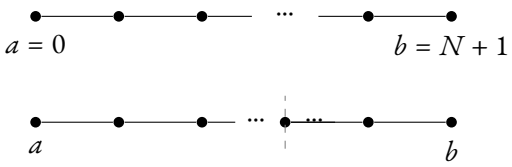


Figure 6.1 Top: a finite path that satisfies $\delta_0, \ldots, \delta_N$, since a and b are $N + 1$ steps apart. Bottom: a model of $\Gamma \cup \{\delta_n : n \geq 0\}$, in which a and b lie in different pieces, so it is not connected and cannot satisfy X.

***Exercise** 6.1.2* Write out δ_0 and δ_1, and check that the path in Figure 6.1 satisfies them when $N \geq 1$.

By Theorem 6.1.1, no set of sentences, and in particular no single sentence, defines connectivity. No single sentence defines connectivity on finite graphs either. Compactness cannot prove this, because the models it produces may be infinite. The usual proof uses Ehrenfeucht–Fraïssé games and is beyond the course.[1]

***Exercise** 6.1.3* A graph is *bipartite* if its vertices can be split into two sets with every edge going between them; equivalently, it has no closed walk of odd length (a walk from a vertex back to itself). Show that the bipartite graphs are the models of a set of sentences, but not of a single sentence. (If φ worked, consider $\Gamma \cup \{\neg\varphi\}$ together with sentences saying there is no closed walk of odd length at most n.)

[1] R. Fraïssé, "Sur quelques classifications des systèmes de relations", *Publications Scientifiques de l'Université d'Alger, Série A* 1 (1954), 35–182; A. Ehrenfeucht, "An application of games to the completeness problem for formalized theories", *Fundamenta Mathematicae* 49 (1961), 129–141. In such a game two players compare two structures move by move; the second player has a winning strategy in the n-move game if and only if the structures satisfy the same sentences with at most n nested quantifiers.

***Exercise** 6.1.4* Show that no set of sentences of L has as its models exactly the graphs in which every vertex has finitely many neighbours. (Add constants $c, d_1, d_2, \ldots$ with $E(c, d_i)$ for all i and $\neg(d_i \approx d_j)$ for $i \neq j$.)

In the next exercises, a *cycle of length* $n \geq 3$ in a graph is a sequence of distinct vertices $s_1, \ldots, s_n$ with s_i joined to s_{i+1} for $i < n$ and s_n joined to s_1. A graph is *acyclic* if it has no cycle, and a *tree* if it is connected and acyclic.

***Exercise** 6.1.5* Let K be a class of graphs. Suppose that there are sets Y and Y' of sentences such that a graph is in K if and only if it satisfies Y, and is outside K if and only if it satisfies Y'. Show that there is a single sentence φ such that a graph is in K if and only if it satisfies φ. (The set $\Gamma \cup Y \cup Y'$ is not satisfiable, so by compactness some finite subset is not; let φ be the conjunction of the sentences of Y in that subset.)

***Exercise** 6.1.6* Show that the acyclic graphs are the models of a set of sentences but not of a single sentence. Using Exercise 6.1.5, conclude that no set of sentences has as its models exactly the graphs that contain a cycle.

***Exercise** 6.1.7* Show that no set of sentences has as its models exactly the trees. (Adapt the proof of Theorem 6.1.1.)

6.2 Nonstandard models of arithmetic

Let L_A have a constant 0, a unary function symbol S (successor), binary function symbols + and ·, and a binary relation symbol <. The *standard model* $\mathbb{N}$ has the natural numbers as elements, with the usual meanings. For $n \geq 0$, the *numeral* $\underline{n}$ is the closed term $S(S(\cdots S(0)\cdots))$ with n copies of S; its value in $\mathbb{N}$ is n. Let $\mathrm{Th}(\mathbb{N})$ be the set of all sentences of L_A that are true in $\mathbb{N}$.

Theorem 6.2.1 (Skolem, 1934) *There is a structure M with $M \models \mathrm{Th}(\mathbb{N})$ and an element s with $\underline{n}^M <^M s$ for every n.*

Proof. Add a new constant c and let $X^+ = \mathrm{Th}(\mathbb{N}) \cup \{\underline{n} < c : n \geq 0\}$. A finite subset of X^+ mentions $\underline{n} < c$ only for $n \leq N$, for some N. Interpret c in $\mathbb{N}$ as $N + 1$. Then $\mathbb{N}$ satisfies $\mathrm{Th}(\mathbb{N})$, and for $n \leq N$ the numeral $\underline{n}$ has value $n < N + 1$, so $\mathbb{N}$ satisfies the finite subset. By Theorem 4.2.10, X^+ has a model; let s be the value of c in it. Forgetting c gives a structure M with the same elements, which satisfies $\mathrm{Th}(\mathbb{N})$ by Lemma 4.2.6(3). Every sentence $\underline{n} < c$ holds in the model of X^+, so $\underline{n}^M <^M s$ for every n.

Every element of $\mathbb{N}$ is the value of a numeral, so no element of $\mathbb{N}$ lies above all numerals. Hence $\mathrm{Th}(\mathbb{N})$ does not single out $\mathbb{N}$. A model of $\mathrm{Th}(\mathbb{N})$ with an element above every $\underline{n}^M$ is a *nonstandard model of arithmetic*, and the values $\underline{n}^M$ are its *standard* elements. The language of X^+ is countable, so by applying the downward Löwenheim–Skolem

theorem to X^+ we may take M countable. Thoralf Skolem built the first nonstandard model in 1934, without compactness.[2]

The order of a nonstandard model Since every sentence true in $\mathbb{N}$ holds in M, the order of M is as follows.

- The values $\underline{n}^M$ form a copy of $\mathbb{N}$ at the start of M. The sentences $\underline{m} < \underline{n}$ for $m < n$ are true in $\mathbb{N}$, and so are $\forall x\, \neg(x < x)$ and $\forall x\, \neg(x < \underline{0})$. For $n \geq 1$, so is $\forall x\, (x < \underline{n} \supset x \approx \underline{0} \vee \ldots \vee x \approx \underline{n-1})$. Hence the $\underline{n}^M$ are distinct and increasing, and nothing in M lies below $\underline{n}^M$ except $\underline{0}^M, \ldots, \underline{n-1}^M$.
- The other elements are *nonstandard*, and each lies above every standard element. The sentences saying that $<$ is irreflexive, transitive and total are true in $\mathbb{N}$, so $<^M$ is a linear order. For nonstandard d and any n, the first item rules out $d <^M \underline{n}^M$, and $d \neq \underline{n}^M$ by definition. Since $<^M$ is total, $\underline{n}^M <^M d$.
- Every nonzero element of $\mathbb{N}$ has a predecessor. A nonstandard d is not $\underline{0}^M$, so it has a predecessor, an element $d-1$ with $S^M(d-1) = d$. This predecessor is again nonstandard, since $d - 1 = \underline{n}^M$ would give $d = \underline{n+1}^M$. Write $d+1$ for $S^M(d)$; it is greater than d, since $\forall x\, (x < S(x))$ is true in $\mathbb{N}$. It is also nonstandard, since $d+1 = \underline{n}^M$ would put d below $\underline{n}^M$. The sentence $\forall x\, \forall y\, \neg(x < y \wedge y < S(x))$ is true in $\mathbb{N}$, so nothing lies between d and $d+1$. Hence the elements $\ldots, d-1, d, d+1, \ldots$ form a copy of $\mathbb{Z}$, called the *block* of d.
- There is no first block and no last block, and between any two blocks there is a third (Exercise 6.2.2). Figure 6.2 shows the whole order.

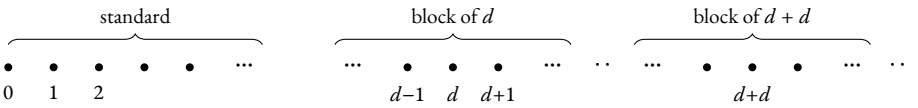


Figure 6.2 A nonstandard model of arithmetic: the standard numbers, then blocks that each look like $\mathbb{Z}$. Between any two blocks there is another, and there is no last block.

***Exercise** 6.2.2* Let $d < d'$ be nonstandard and in different blocks. Using sentences true in $\mathbb{N}$, show that $d + d$ is not in the block of d, that some nonstandard element lies below the block of d, and that some element lies strictly between the blocks of d and d'. (Every element of $\mathbb{N}$ is of the form $y + y$ or $y + y + 1$; halve d, and halve $d + d'$.)

***Exercise** 6.2.3* Show that no formula $\varphi(x)$ of L_A defines the standard elements, in the sense that $(M, \sigma[x \mapsto s]) \vDash \varphi(x)$ exactly when s is standard. (If it did, the sentence $(\varphi(0) \wedge \forall x\, (\varphi(x) \supset \varphi(S(x)))) \supset \forall x\, \varphi(x)$, which is true in $\mathbb{N}$, would hold in M. Here $\varphi(S(x))$ is the formula obtained from φ by replacing each free x by $S(x)$; assume x is not bound in φ.)

[2] T. Skolem, "Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen", *Fundamenta Mathematicae* 23 (1934), 150–161. His elements were definable functions from $\mathbb{N}$ to $\mathbb{N}$, compared on a suitably chosen infinite set of arguments. Jerzy Łoś made this construction, the *ultrapower*, a general tool in 1955.