METHODS FOR EFFICIENT SAT SOLVING

Aalok Thakkar

Indian Statistical Institute, Kolkata

October 28, 2024

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Our lab is offering PhD, visiting researcher, and internship positions.
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$$\{A \vee B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

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What are $\vee$, $\leftarrow$, $\rightarrow$, $\neg$, $\oplus$, $\leftrightarrow$?

What are variables (atoms)?

What are assignments (models)?

What does satisfaction mean?

$$\{A \vee B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

$$\{A \vee B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

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Is there a satisfying assignment?

$$\{A \vee B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

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Is there a satisfying assignment?

Check all assignments!

$$\{A \vee B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

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Is there a satisfying assignment?

Check all assignments!

Can we do better?

$$\{A \vee B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

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Is there a satisfying assignment?

Check all assignments!

Can we do something smarter?

Conjunctive Normal Form (CNF)

$$\{A \vee B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

$$\{A \vee B, A \vee \neg(\neg C \vee D), \neg((D \vee A) \wedge \neg(D \wedge A)), (B \wedge D) \vee (\neg B \wedge \neg D)\}$$

Conjunctive Normal Form (CNF)

$$\{A \lor B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

$$\{A \lor B, A \lor \neg(\neg C \lor D), \neg((D \lor A) \land \neg(D \land A)), (B \land D) \lor (\neg B \land \neg D)\}$$

$$p = c_1 \land c_2 \land \ldots \land c_m$$

$$c_i = l_{i,1} \lor l_{i,2} \lor \ldots \lor l_{i,n_i}$$

$l_{i,j} = A$ or $l_{i,j} = \neg A$

Conjunctive Normal Form (CNF)

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$$l_{i,j} = A \text{ or } l_{i,j} = \neg A$$

Normalisation Theorem: For every propositional formula, there exists an *equivalent* formula in conjunctive normal form.

Conjunctive Normal Form (CNF)

$$\{A \vee B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

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Tseitin's Theorem: For every propositional formula, there exists a <u>polynomial size</u> *equisatisfiable* formula in conjunctive normal form.

Conjunctive Normal Form (CNF)

$$\{A \vee B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

$$\{A \vee B, A \vee \neg(\neg C \vee D), \neg((D \vee A) \wedge \neg(D \wedge A)), (B \wedge D) \vee (\neg B \wedge \neg D)\}$$

$A \vee B$

Conjunctive Normal Form (CNF)

$$\{A \lor B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

$$\{A \lor B, A \lor \neg(\neg C \lor D), \neg((D \lor A) \land \neg(D \land A)), (B \land D) \lor (\neg B \land \neg D)\}$$

$A \lor B$

$A \lor \neg(\neg C \lor D)$

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$$\{A \vee B, A \vee \neg(\neg C \vee D), \neg((D \vee A) \wedge \neg(D \wedge A)), (B \wedge D) \vee (\neg B \wedge \neg D)\}$$

$$A \vee B$$

$$A \vee \neg(\neg C \vee D) \implies A \vee (C \wedge \neg D) \implies (A \vee C) \wedge (A \vee \neg D)$$

Conjunctive Normal Form (CNF)

$$\{A \lor B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

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$$A \lor B$$

$$A \lor \neg(\neg C \lor D) \implies A \lor (C \land \neg D) \implies (A \lor C) \land (A \lor \neg D)$$

$$\neg((D \lor A) \land \neg(D \land A))$$

Conjunctive Normal Form (CNF)

$$\{A \vee B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

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$A \vee B$

$$A \vee \neg(\neg C \vee D) \implies A \vee (C \wedge \neg D) \implies (A \vee C) \wedge (A \vee \neg D)$$

$$\neg((D \vee A) \wedge \neg(D \wedge A)) \implies (\neg D \wedge \neg A) \vee (D \wedge A) \implies$$

$$((\neg D \wedge \neg A) \vee D) \wedge ((\neg D \wedge \neg A) \vee A) \implies$$

$$(\neg D \vee D) \wedge (\neg A \vee D) \wedge (\neg D \vee A) \wedge (\neg A \vee A)$$

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$$\neg((D \vee A) \wedge \neg(D \wedge A)) \implies (\neg D \wedge \neg A) \vee (D \wedge A) \implies$$

$$((\neg D \wedge \neg A) \vee D) \wedge ((\neg D \wedge \neg A) \vee A) \implies$$

$$(\neg D \vee D) \wedge (\neg A \vee D) \wedge (\neg D \vee A) \wedge (\neg A \vee A)$$

$$\ldots$$

Conjunctive Normal Form (CNF)

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$$\{(A \vee B), (A \vee C), (A \vee \neg D), (\neg A \vee D), (\neg D \vee A), (B \vee \neg D), (D \vee \neg B)\}$$

Can find a satisfying assignment in polynomial time?

Conjunctive Normal Form (CNF)

$$\{A \vee B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

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$$\{(A \vee B), (A \vee C), (A \vee \neg D), (\neg A \vee D), (\neg D \vee A), (B \vee \neg D), (D \vee \neg B)\}$$

$$\neg A \rightarrow B$$

$$\neg A \rightarrow C$$

$$D \rightarrow A$$

$$A \rightarrow D$$

$$B \rightarrow D$$

$$D \rightarrow B$$

Conjunctive Normal Form (CNF)

$$\{A \vee B, A \leftarrow (C \rightarrow D), \neg(D \oplus A), B \leftrightarrow D\}$$

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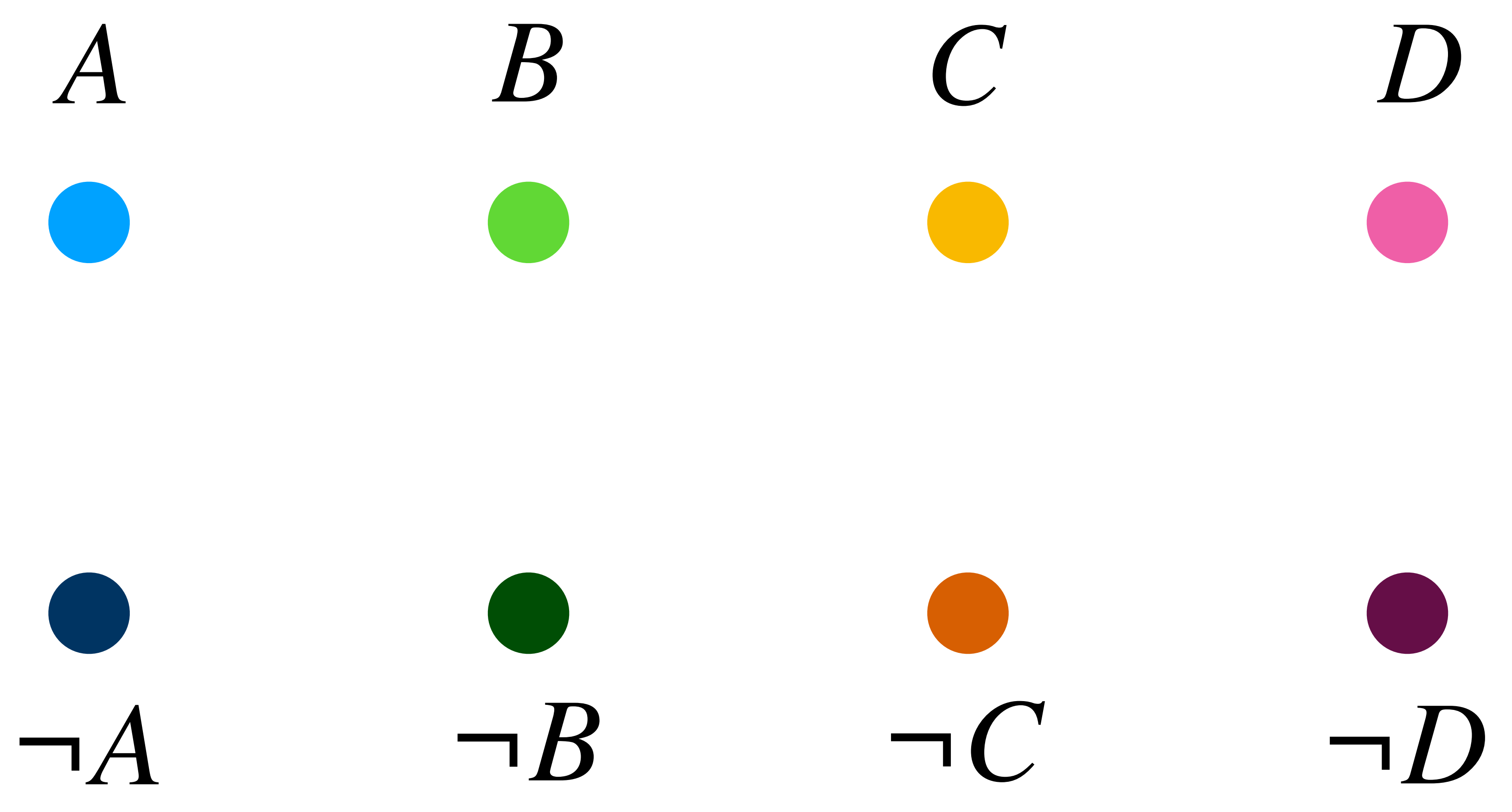
$$\{(A \vee B), (A \vee C), (A \vee \neg D), (\neg A \vee D), (\neg D \vee A), (B \vee \neg D), (D \vee \neg B)\}$$

$\neg A \rightarrow B$	$\neg B \rightarrow A$
$\neg A \rightarrow C$	$\neg C \rightarrow A$
$D \rightarrow A$	$\neg A \rightarrow \neg D$
$A \rightarrow D$	$\neg D \rightarrow \neg A$
$B \rightarrow D$	$\neg D \rightarrow \neg B$
$D \rightarrow B$	$\neg B \rightarrow \neg D$



$\neg A \rightarrow B$ $\qquad$ $\neg B \rightarrow A$

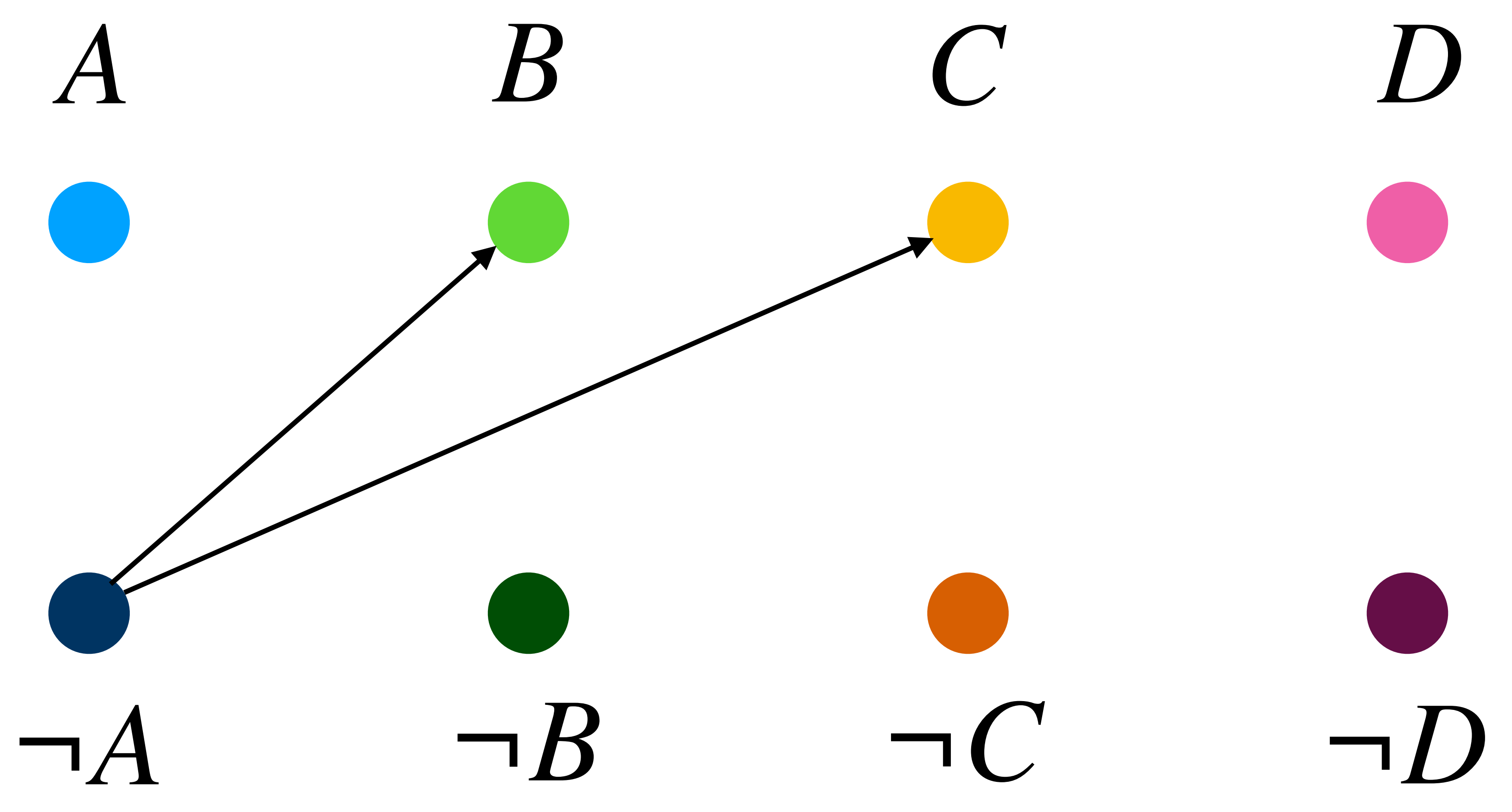
$\neg A \rightarrow C$ $\qquad$ $\neg C \rightarrow A$

$D \rightarrow A$ $\qquad$ $\neg A \rightarrow \neg D$

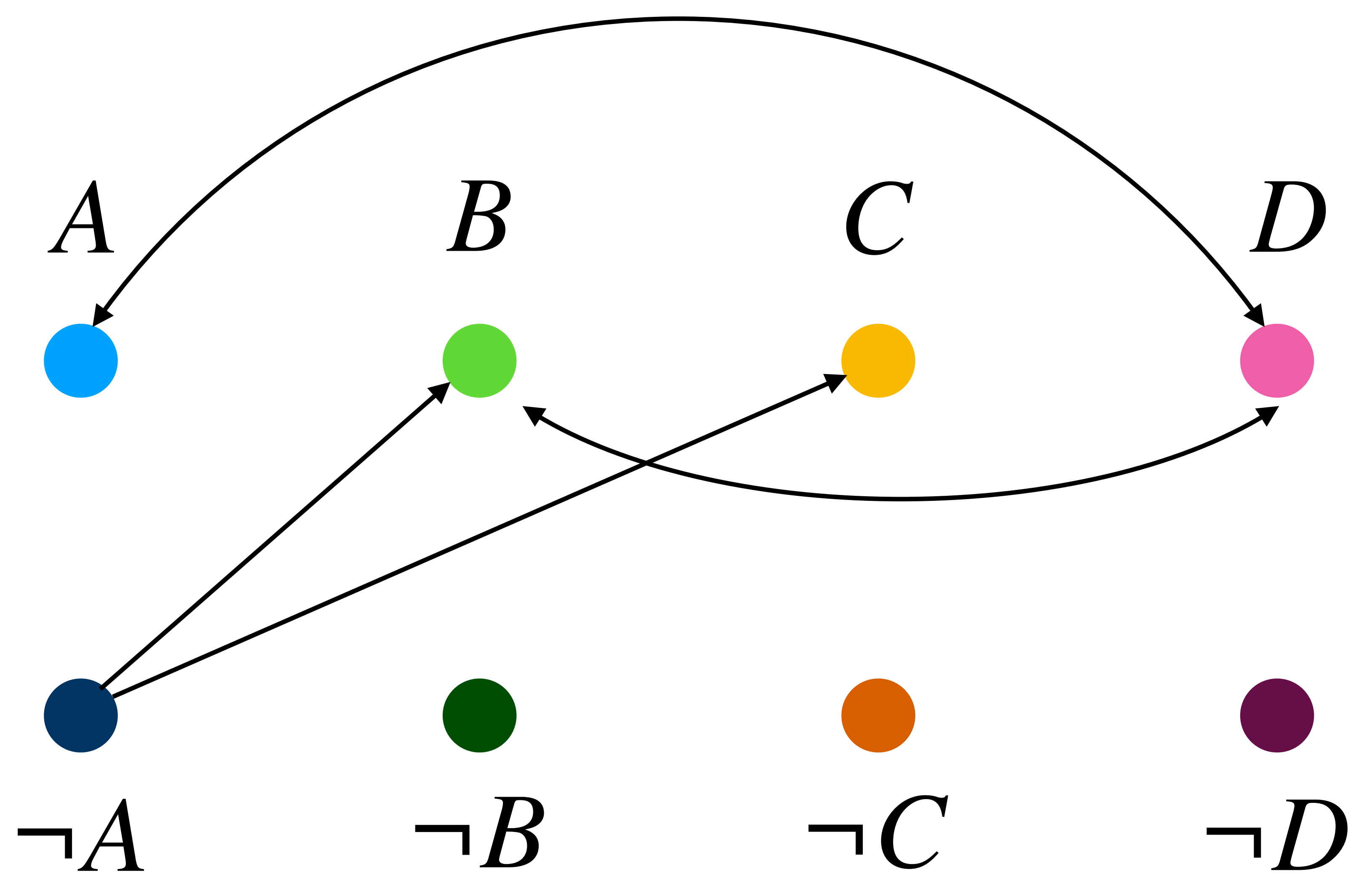
$A \rightarrow D$ $\qquad$ $\neg D \rightarrow \neg A$

$B \rightarrow D$ $\qquad$ $\neg D \rightarrow \neg B$

$D \rightarrow B$ $\qquad$ $\neg B \rightarrow \neg D$



$\neg A \to B$	$\neg B \to A$
$\neg A \to C$	$\neg C \to A$
$D \to A$	$\neg A \to \neg D$
$A \to D$	$\neg D \to \neg A$
$B \to D$	$\neg D \to \neg B$
$D \to B$	$\neg B \to \neg D$



$\neg A \rightarrow B$	$\neg B \rightarrow A$
$\neg A \rightarrow C$	$\neg C \rightarrow A$
$D \rightarrow A$	$\neg A \rightarrow \neg D$
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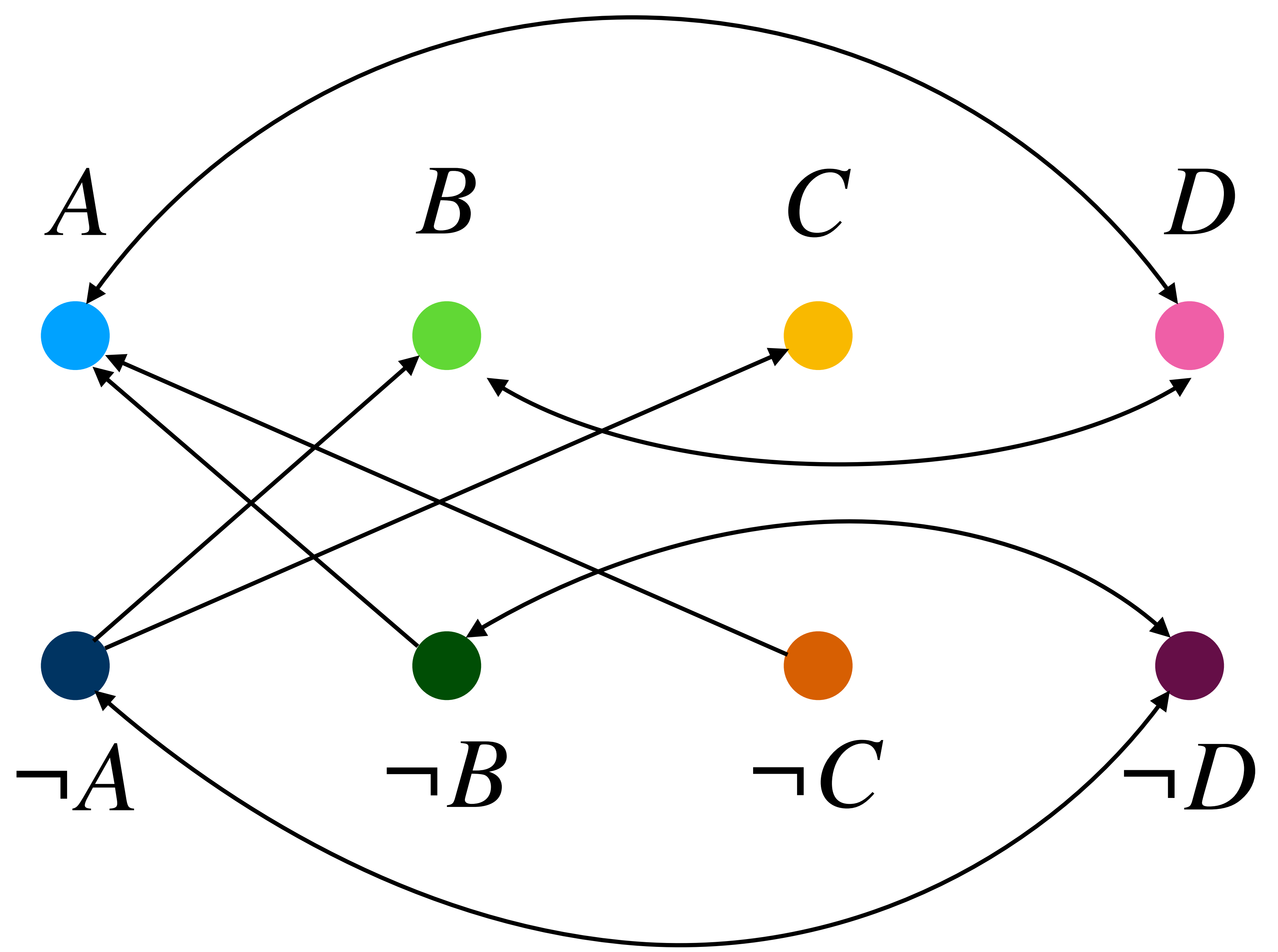


$\neg A \rightarrow B$

$\neg A \rightarrow C$

$D \rightarrow A$

$A \rightarrow D$

$B \rightarrow D$

$D \rightarrow B$

$\neg B \rightarrow A$

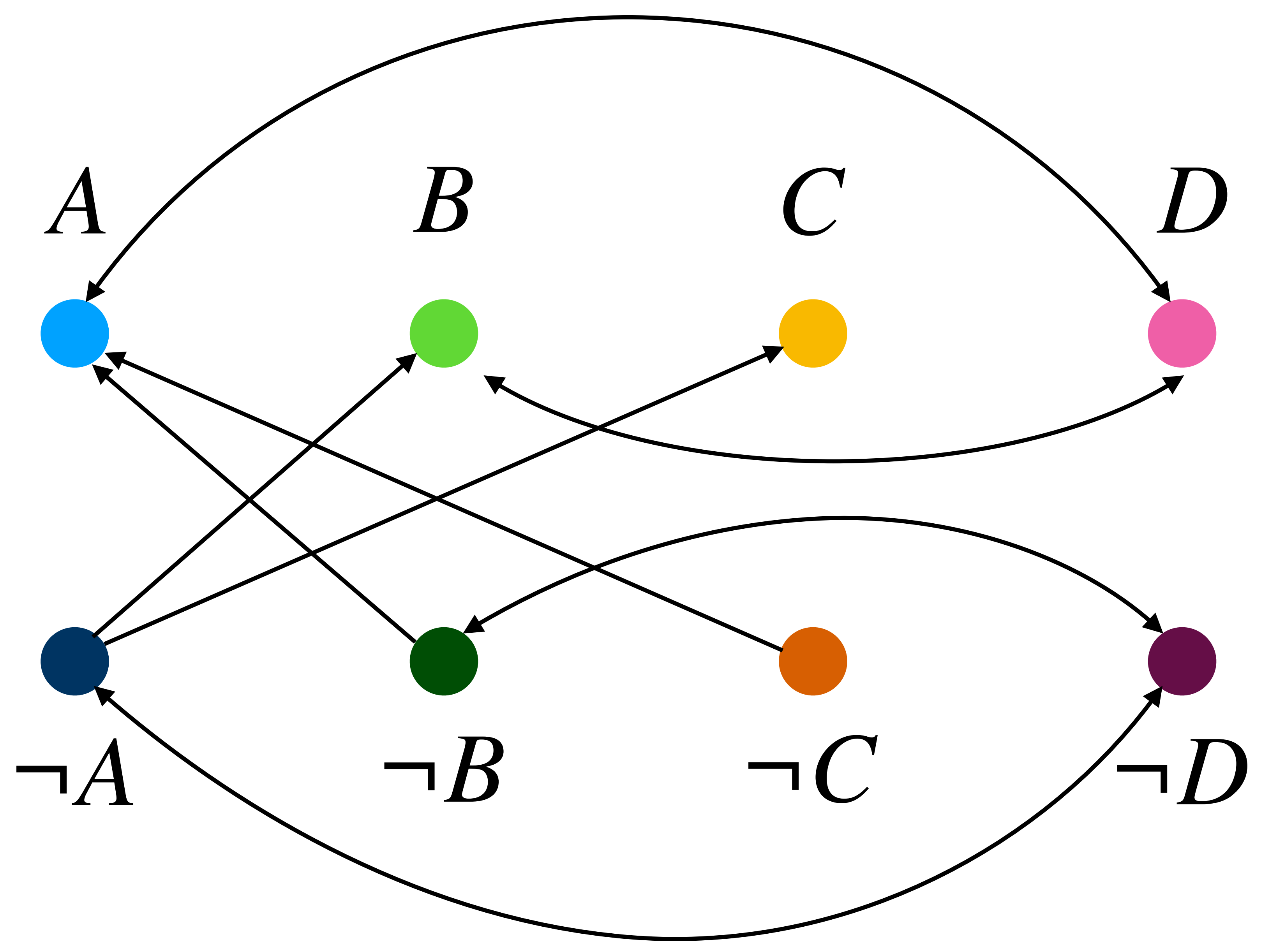
$\neg C \rightarrow A$

$\neg A \rightarrow \neg D$

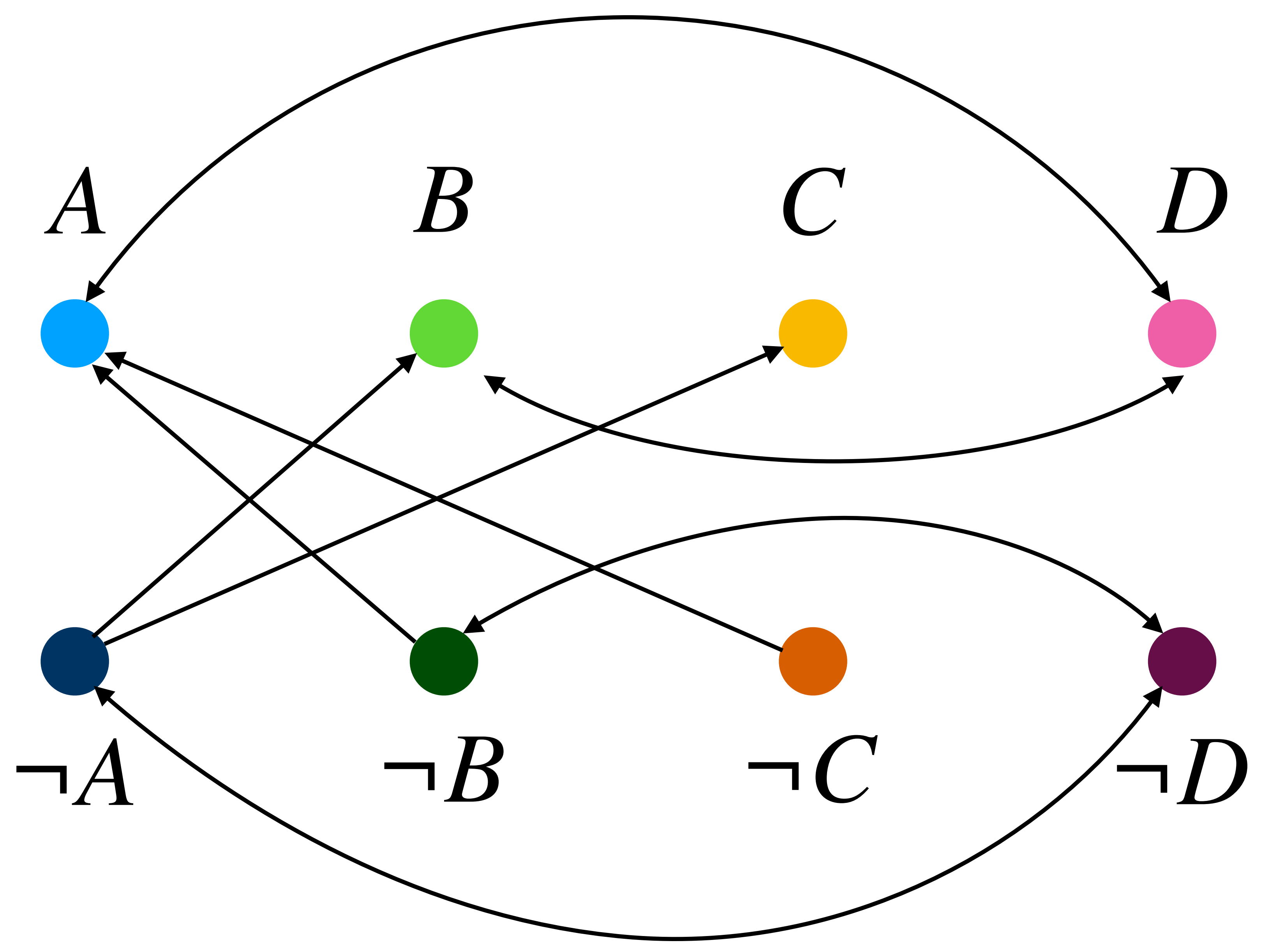
$\neg D \rightarrow \neg A$

$\neg D \rightarrow \neg B$

$\neg B \rightarrow \neg D$



Aspvall, Plass, Tarjan (1979): For any variable X, the vertices for X and $\neg X$ exist in a strongly connected component of the implication graph if and only if the set is not satisfiable.



Aspvall, Plass, Tarjan (1979): For any variable X, the vertices for X and $\neg X$ exist in a strongly connected component of the implication graph if and only if the set is not satisfiable.

When is a SAT problem in P?

Schaefer's dichotomy theorem:

Given a finite set of variables, and a conjunction of constraints, a class of SAT instances is in P if and only if all constraints are:

1. Satisfied when all the variables are true (or when all variables are false).

2. Binary clauses

3. Horn clauses or dual-Horn clauses

4. Affine clauses

What do we want?

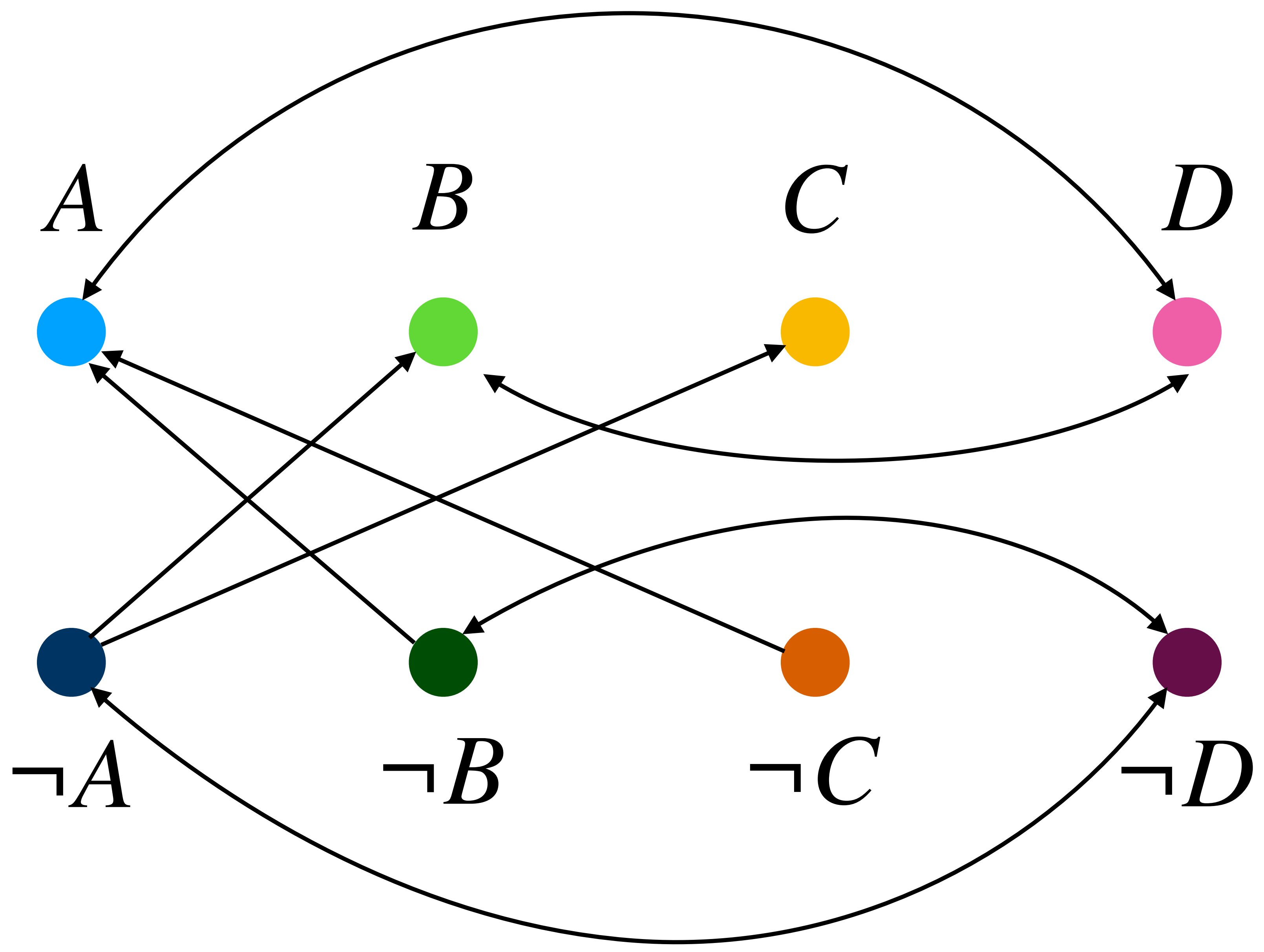
$$A \wedge B \wedge C \rightarrow D$$

$$\texttt{true} \rightarrow A$$

$$D \rightarrow \texttt{false}$$

$$B \wedge C \rightarrow E$$

$$E \wedge C \rightarrow B$$



For certain classes of propositional logic, we have efficient algorithms.

For all of propositional logic, a *somewhat efficient* algorithm?

Davis–Putnam–Logemann–Loveland (DPLL) Algorithm

Partial assignment: $m : \{x_1, \ldots x_n\} \rightarrow \{0,1,?\}$

State of a literal: l is true under m if $m(l) = 1$,
and l is false under m if $m(l) = 0$.

State of a clause: c is true under m if for some $l \in c, m(l) = 1$,
and c is true under m if for all $l \in c, m(l) = 0$.

Unit clause: c is a unit clause under m if exactly one $l \in c$ is unassigned
and the rest are assigned 0. Such an l is called a unit literal.

Davis–Putnam–Logemann–Loveland (DPLL) Algorithm

Input: CNF f, and partial assignment m

Chose an unassigned variable a, and assign it $b \in \{0,1\}$.

If $DPLL\left(f, m[a \to b]\right) = SAT$, return $m[a \to b]$

Else, return $DPLL\left(f, m[a \to 1 - b]\right)$

Davis–Putnam–Logemann–Loveland (DPLL) Algorithm

Input: CNF f, and partial assignment m

If f is true under m, return m.

If f is false under m, return $\perp$.

Chose an unassigned variable a, and assign it $b \in \{0,1\}$.

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Davis–Putnam–Logemann–Loveland (DPLL) Algorithm

Input: CNF f, and partial assignment m

If f is true under m, return m.

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If $\exists$ unit literal p under m, then return $DPLL\left(f, m[p \to 1]\right)$.

Chose an unassigned variable a, and assign it $b \in \{0,1\}$.

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If $\exists$ unit literal $\neg p$ under m, then return $DPLL\left(f, m[p \to 0]\right)$.

Chose an unassigned variable a, and assign it $b \in \{0,1\}$.

If $DPLL\left(f, m[a \to b]\right) = SAT$, return $m[a \to b]$

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$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_5)$$

$$c_3 = (\neg x_2 \vee x_4)$$

$$c_4 = (\neg x_3 \vee \neg x_4)$$

$$c_5 = (x_1 \vee x_5 \vee \neg x_2)$$

$$c_6 = (x_2 \vee x_3)$$

$$c_7 = (x_2 \vee \neg x_3 \vee x_7)$$

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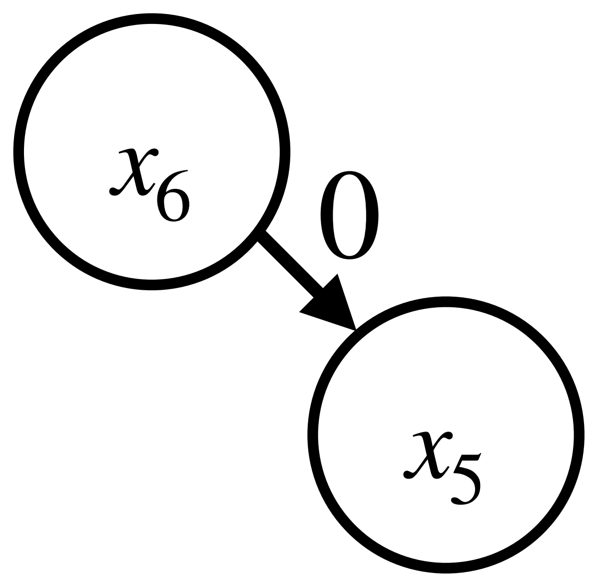

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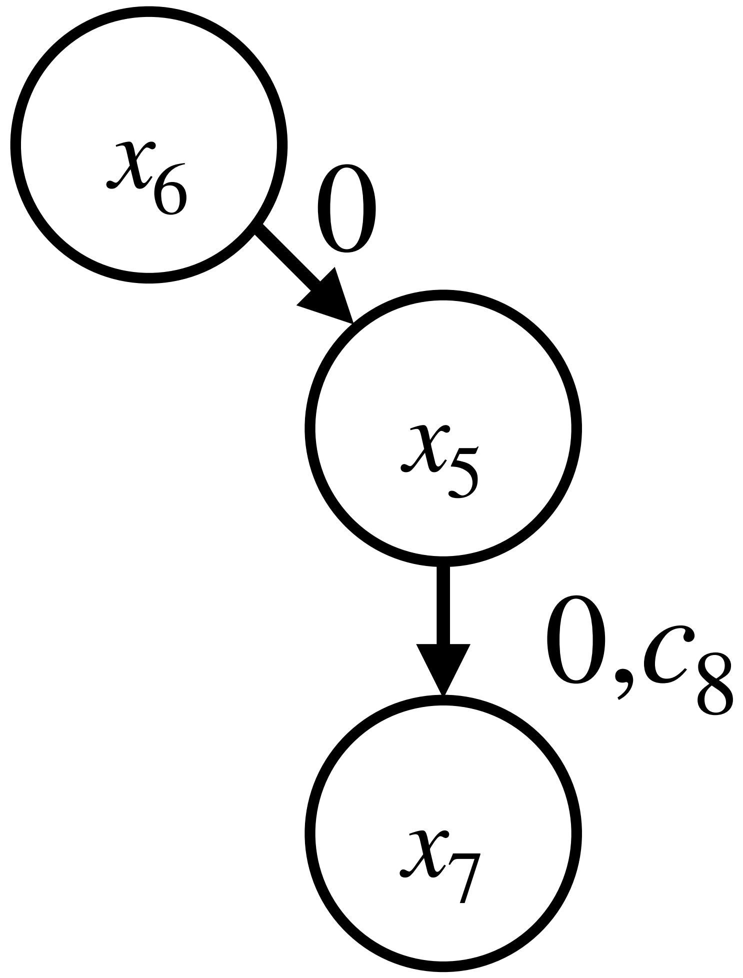

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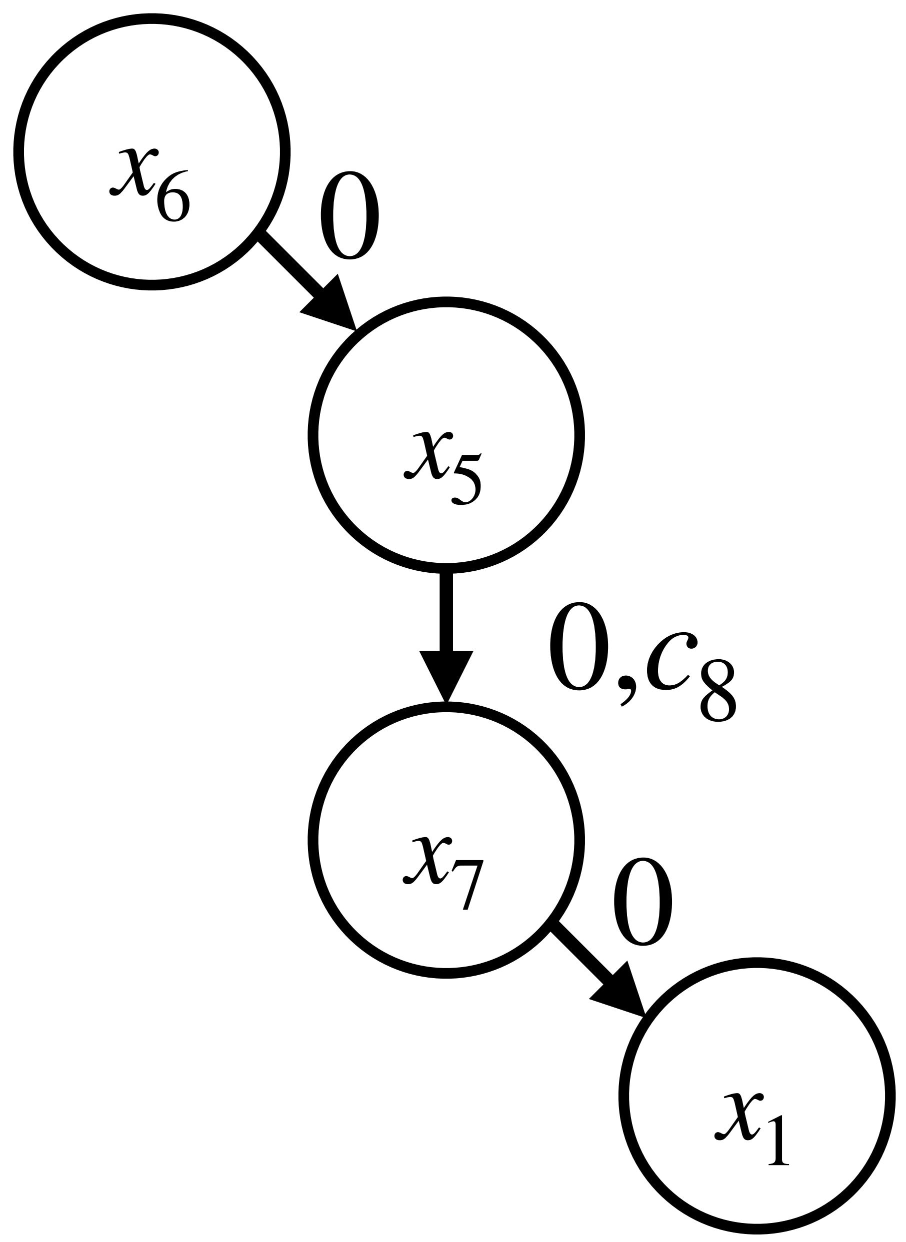

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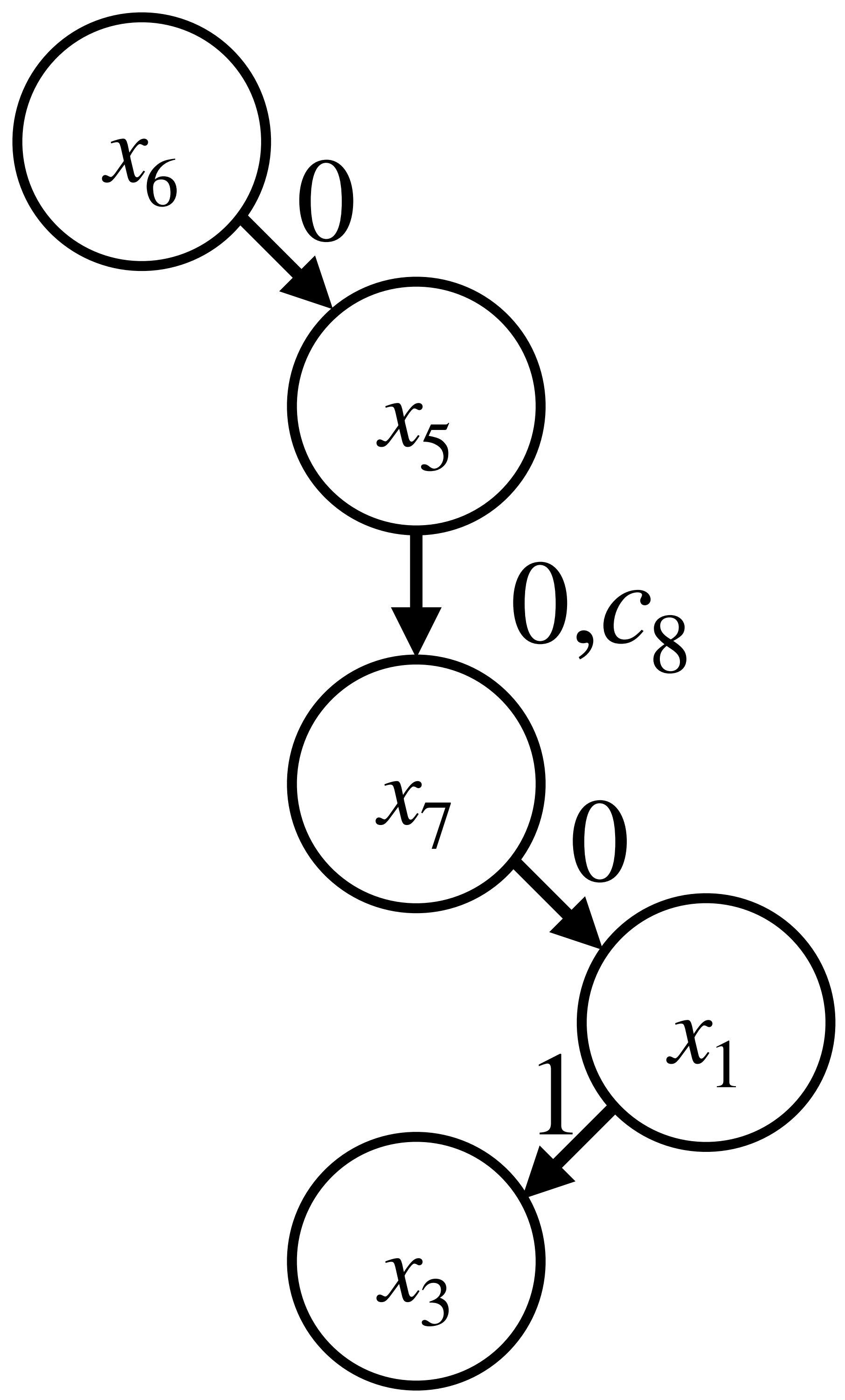

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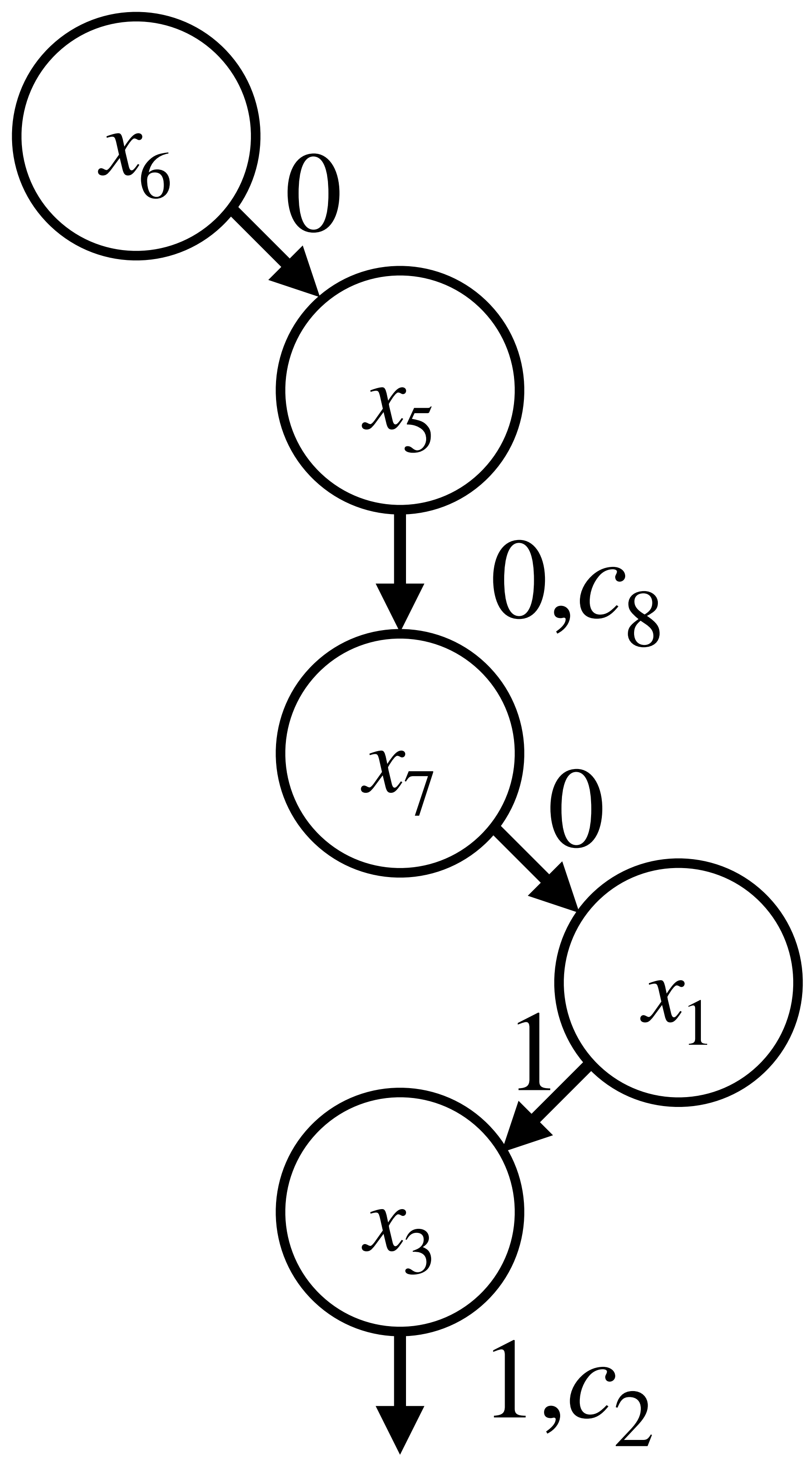

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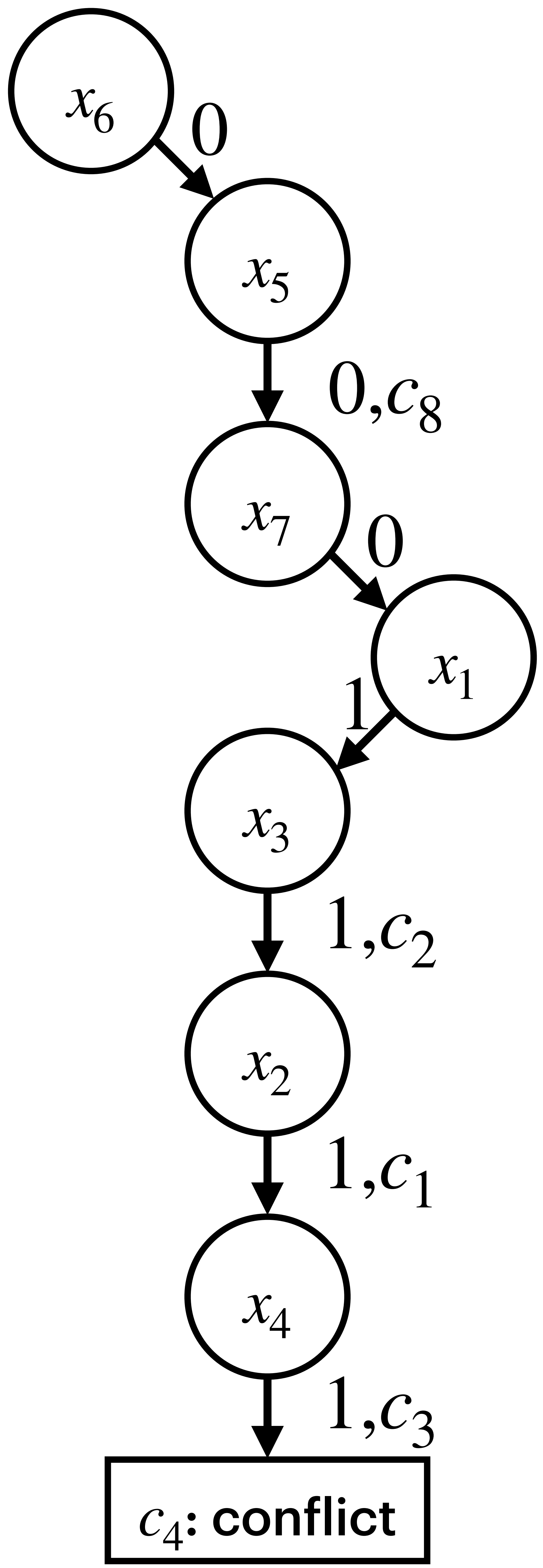

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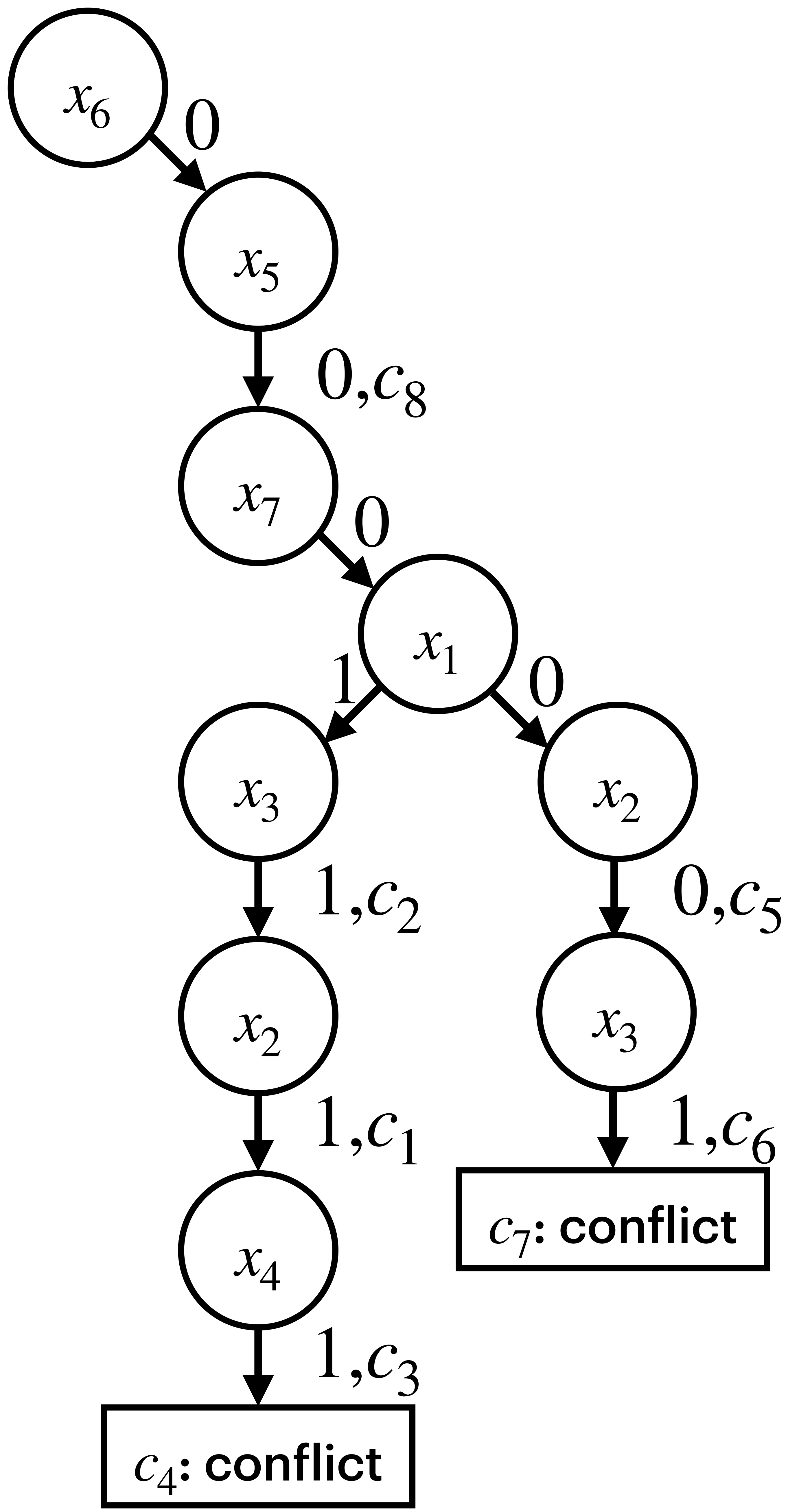
$$c_7 = (x_2 \lor \neg x_3 \lor x_7)$$

$$c_8 = (x_6 \lor \neg x_5)$$



$$c_1 = (\neg x_1 \lor x_2)$$

$$c_2 = (\neg x_1 \lor x_3 \lor x_5)$$

$$c_3 = (\neg x_2 \lor x_4)$$

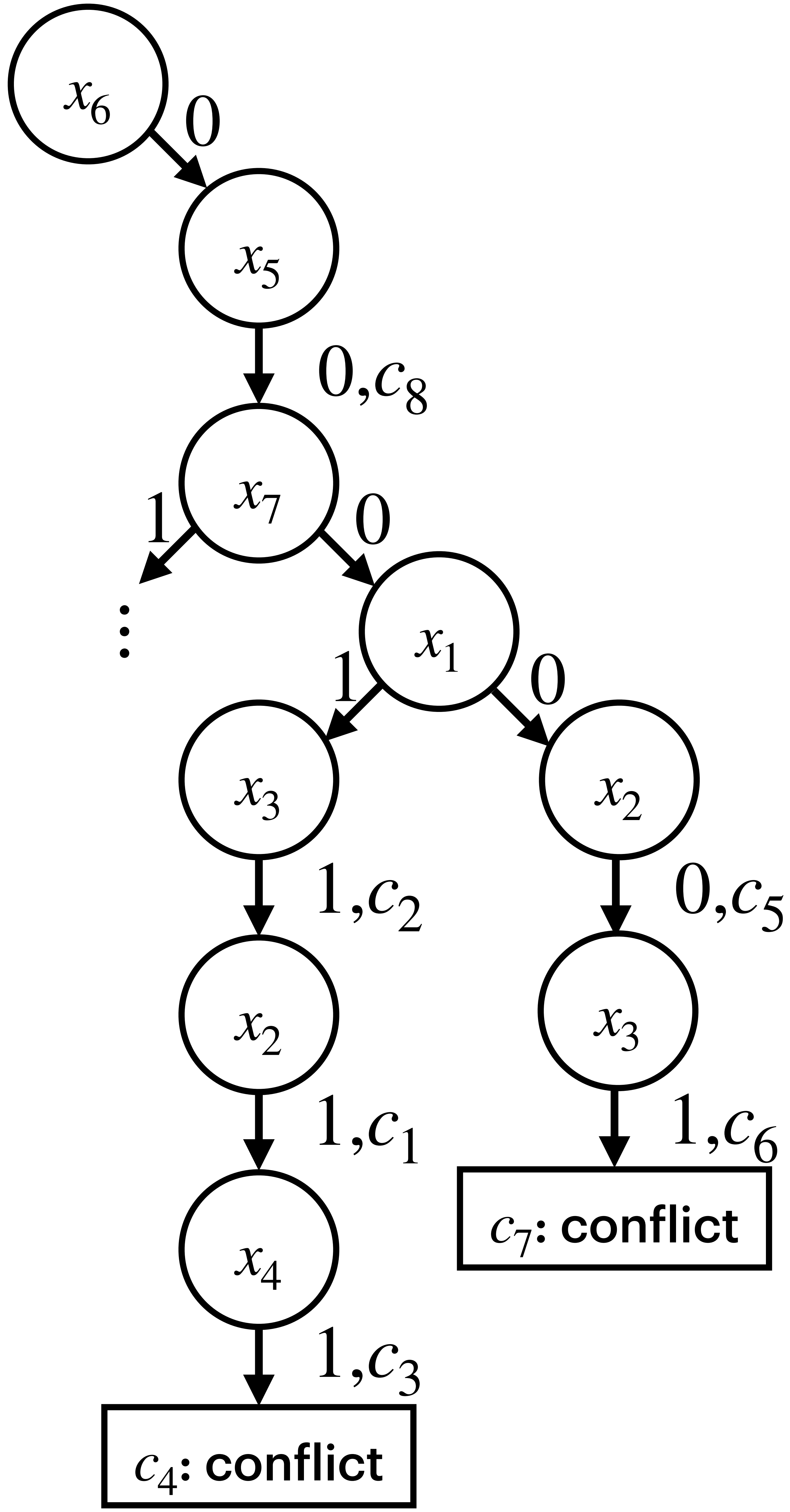
$$c_4 = (\neg x_3 \lor \neg x_4)$$

$$c_5 = (x_1 \lor x_5 \lor \neg x_2)$$

$$c_6 = (x_2 \lor x_3)$$

$$c_7 = (x_2 \lor \neg x_3 \lor x_7)$$

$$c_8 = (x_6 \lor \neg x_5)$$



Time to Code!

$c_1 = (\neg x_1 \vee x_2)$

$c_2 = (\neg x_1 \vee x_3 \vee x_5)$

$c_3 = (\neg x_2 \vee x_4)$

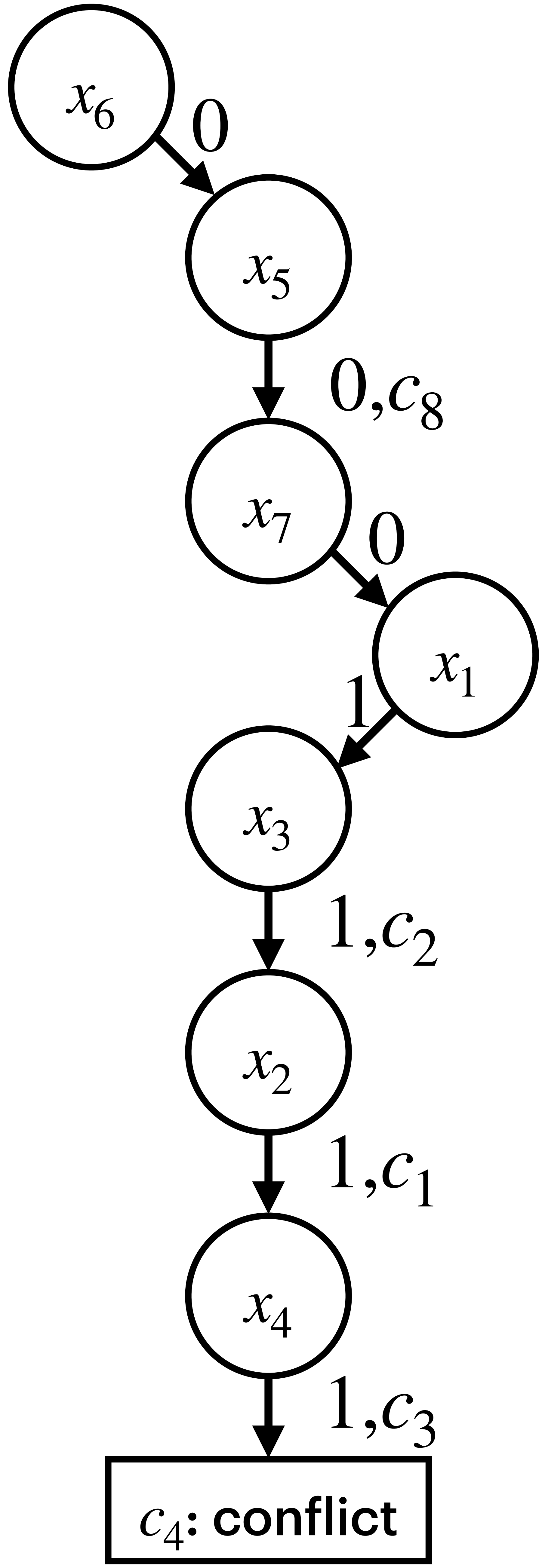
$c_4 = (\neg x_3 \vee \neg x_4)$

$c_5 = (x_1 \vee x_5 \vee \neg x_2)$

$c_6 = (x_2 \vee x_3)$

$c_7 = (x_2 \vee \neg x_3 \vee x_7)$

$c_8 = (x_6 \vee \neg x_5)$



$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_5)$$

$$c_3 = (\neg x_2 \vee x_4)$$

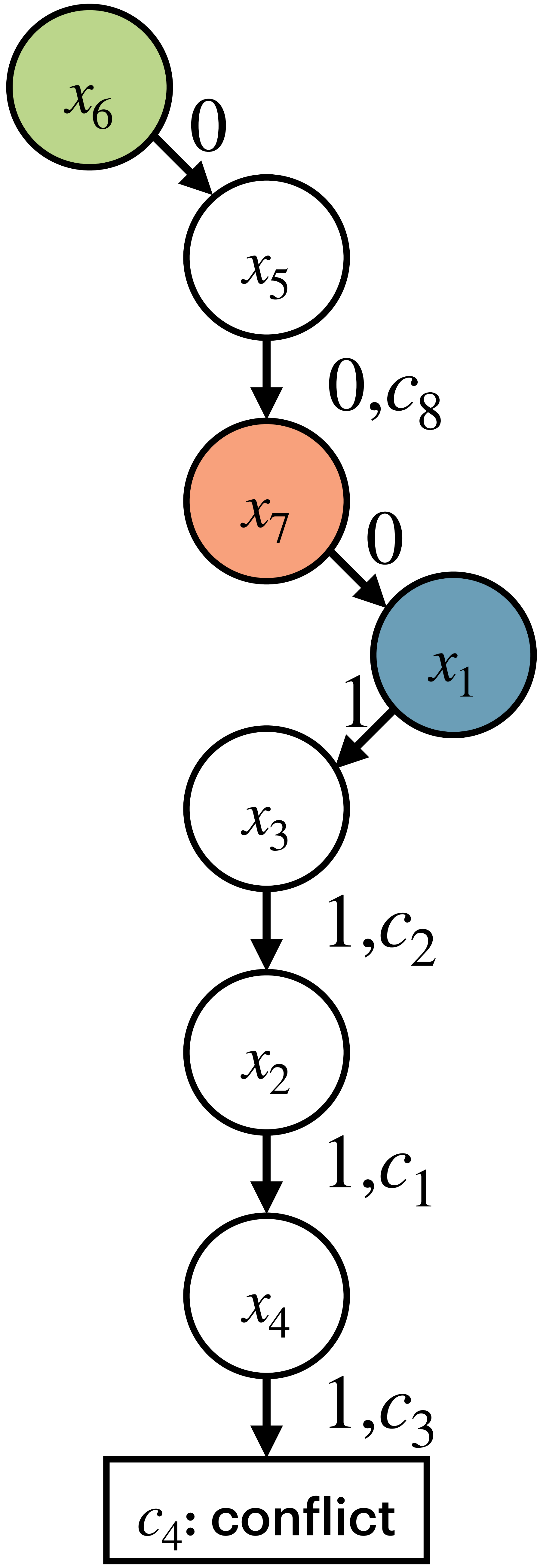
$$c_4 = (\neg x_3 \vee \neg x_4)$$

$$c_5 = (x_1 \vee x_5 \vee \neg x_2)$$

$$c_6 = (x_2 \vee x_3)$$

$$c_7 = (x_2 \vee \neg x_3 \vee x_7)$$

$$c_8 = (x_6 \vee \neg x_5)$$



$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_5)$$

$$c_3 = (\neg x_2 \vee x_4)$$

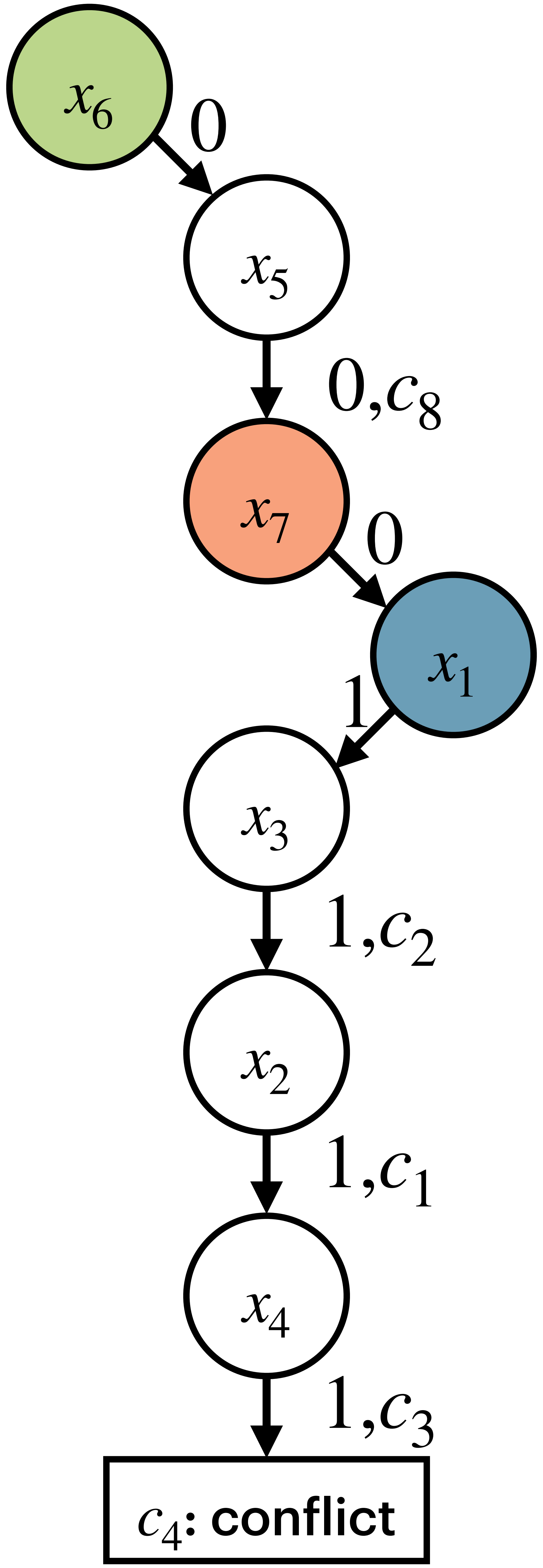
$$c_4 = (\neg x_3 \vee \neg x_4)$$

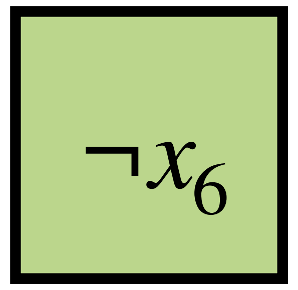
$$c_5 = (x_1 \vee x_5 \vee \neg x_2)$$

$$c_6 = (x_2 \vee x_3)$$

$$c_7 = (x_2 \vee \neg x_3 \vee x_7)$$

$$c_8 = (x_6 \vee \neg x_5)$$





$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_5)$$

$$c_3 = (\neg x_2 \vee x_4)$$

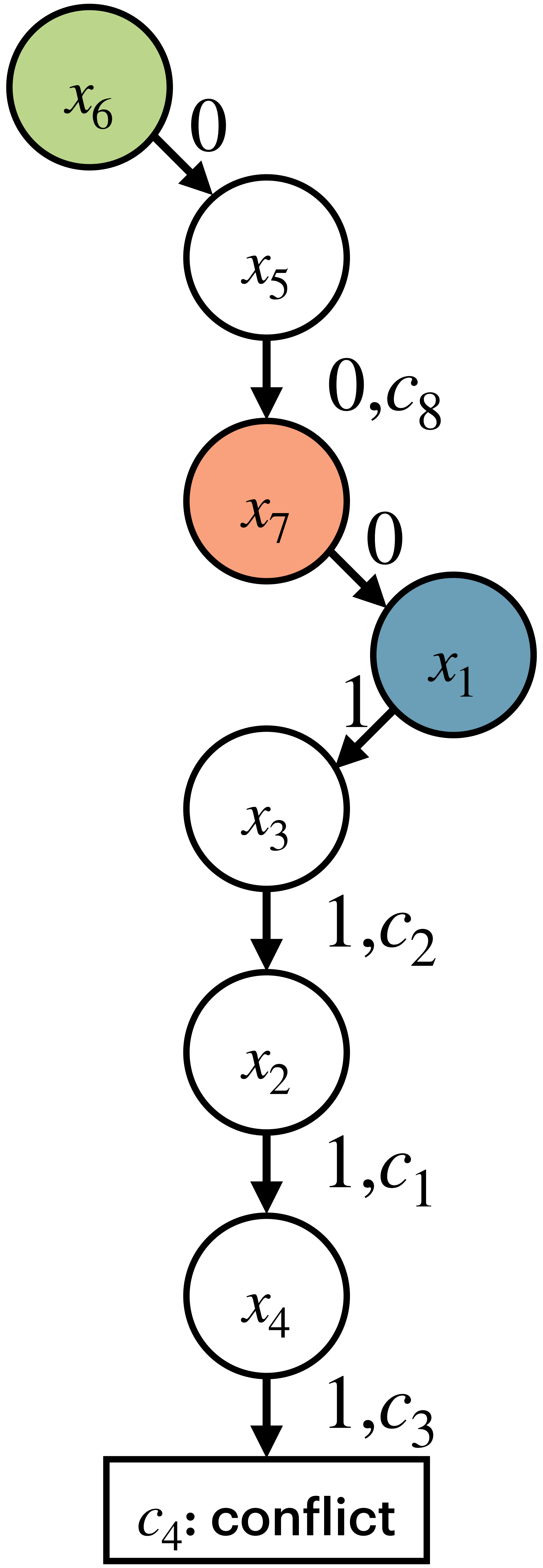
$$c_4 = (\neg x_3 \vee \neg x_4)$$

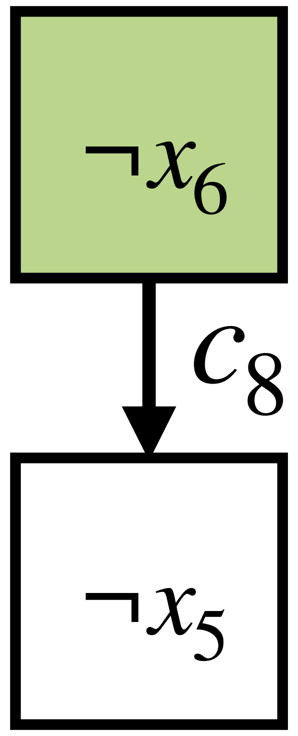
$$c_5 = (x_1 \vee x_5 \vee \neg x_2)$$

$$c_6 = (x_2 \vee x_3)$$

$$c_7 = (x_2 \vee \neg x_3 \vee x_7)$$

$$c_8 = (x_6 \vee \neg x_5)$$





$$c_1 = (\neg x_1 \lor x_2)$$

$$c_2 = (\neg x_1 \lor x_3 \lor x_5)$$

$$c_3 = (\neg x_2 \lor x_4)$$

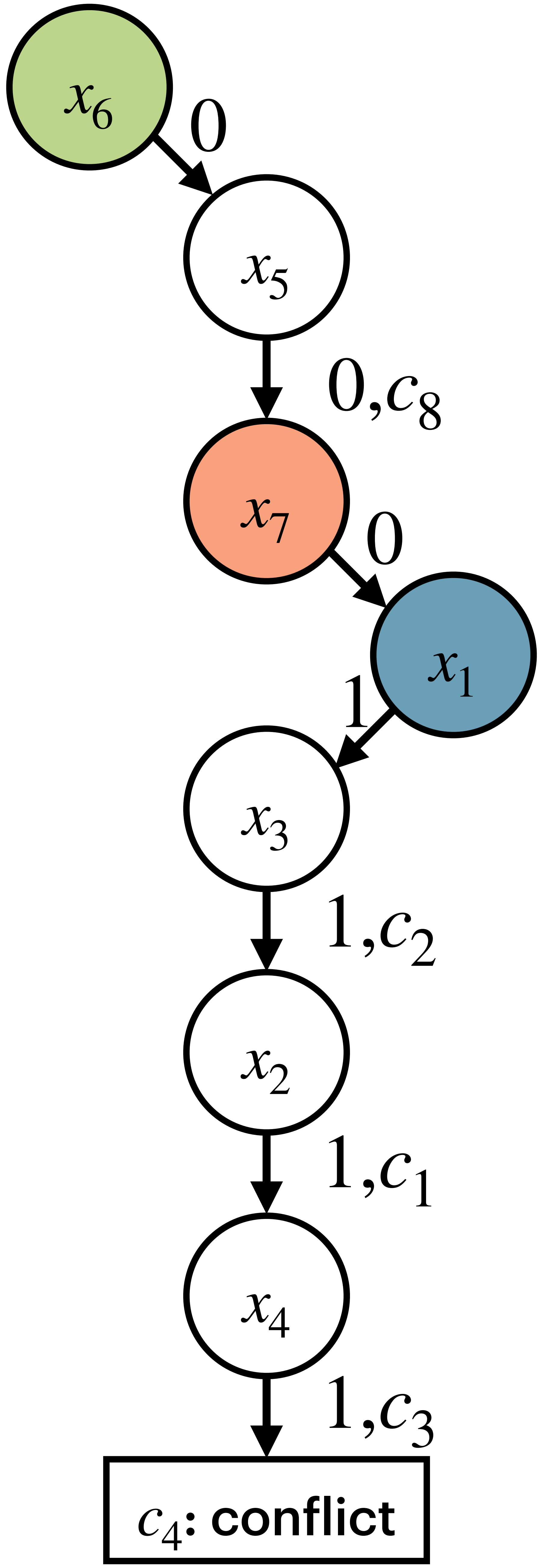
$$c_4 = (\neg x_3 \lor \neg x_4)$$

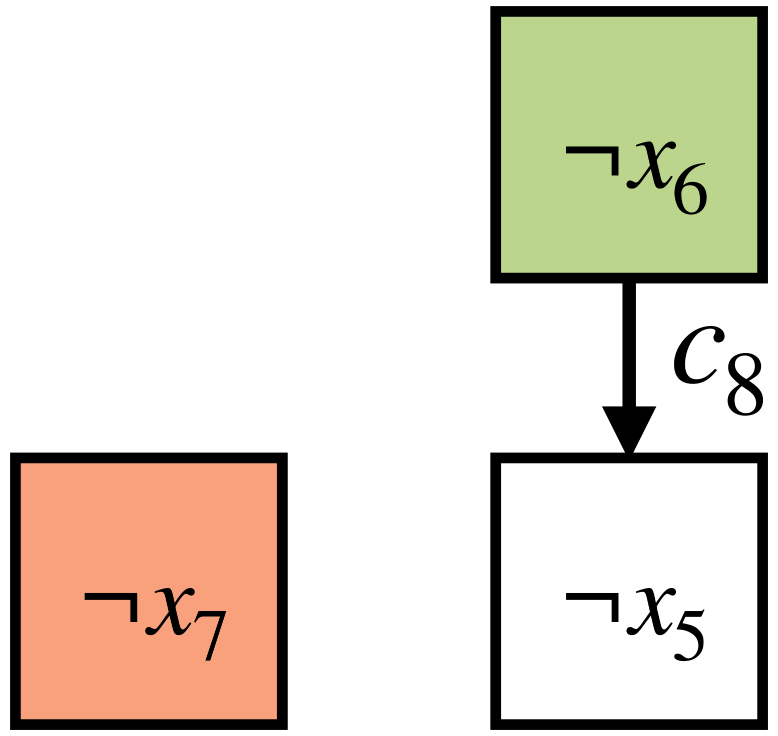
$$c_5 = (x_1 \lor x_5 \lor \neg x_2)$$

$$c_6 = (x_2 \lor x_3)$$

$$c_7 = (x_2 \lor \neg x_3 \lor x_7)$$

$$c_8 = (x_6 \lor \neg x_5)$$





$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_5)$$

$$c_3 = (\neg x_2 \vee x_4)$$

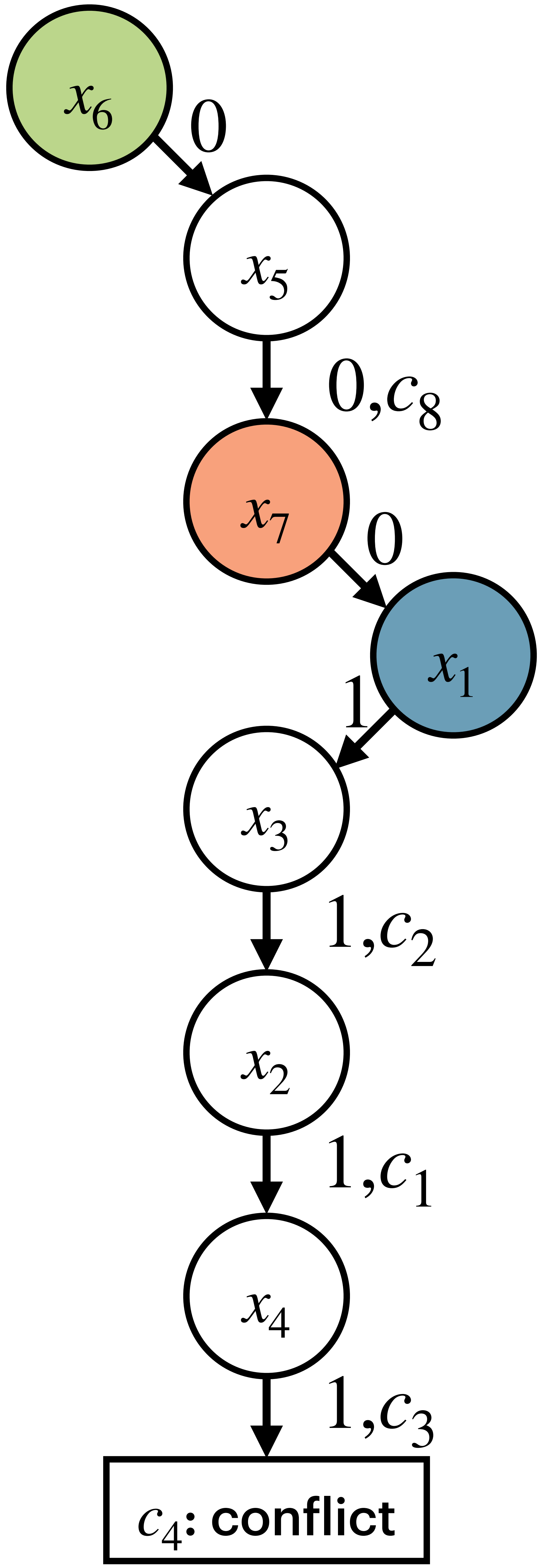
$$c_4 = (\neg x_3 \vee \neg x_4)$$

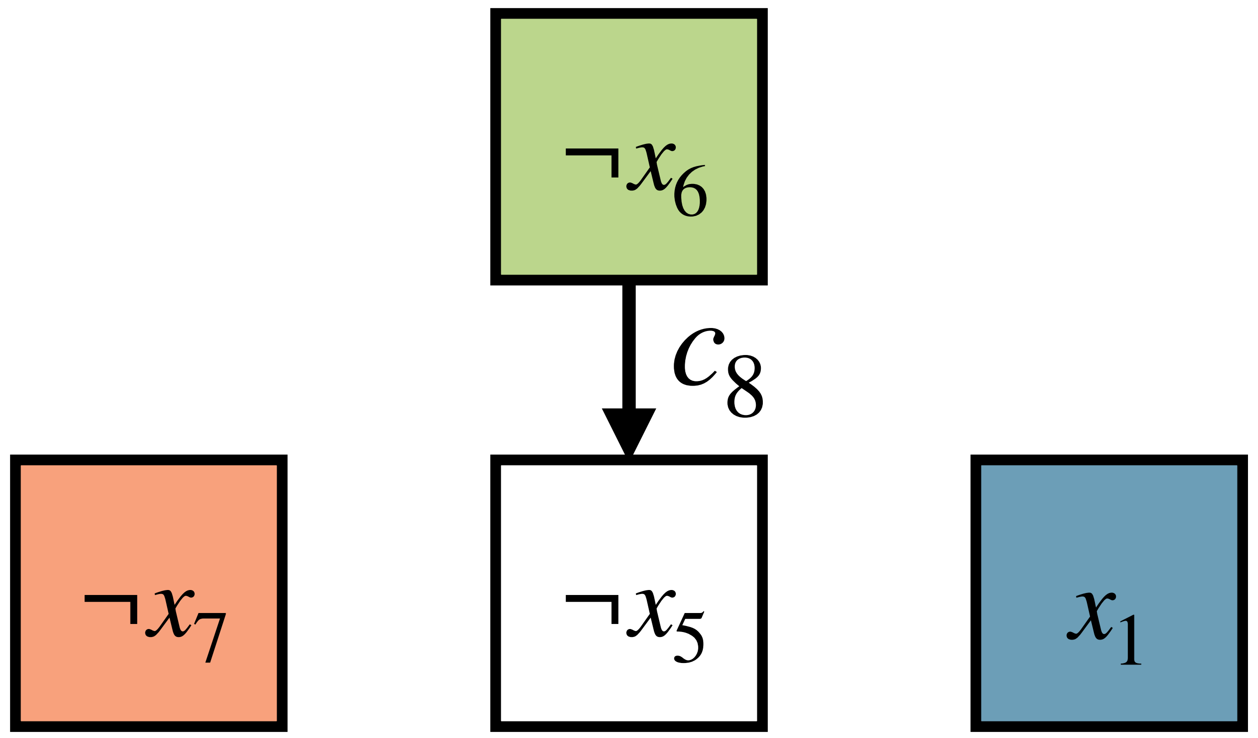
$$c_5 = (x_1 \vee x_5 \vee \neg x_2)$$

$$c_6 = (x_2 \vee x_3)$$

$$c_7 = (x_2 \vee \neg x_3 \vee x_7)$$

$$c_8 = (x_6 \vee \neg x_5)$$





$$c_1 = (\neg x_1 \lor x_2)$$
$$c_2 = (\neg x_1 \lor x_3 \lor x_5)$$
$$c_3 = (\neg x_2 \lor x_4)$$
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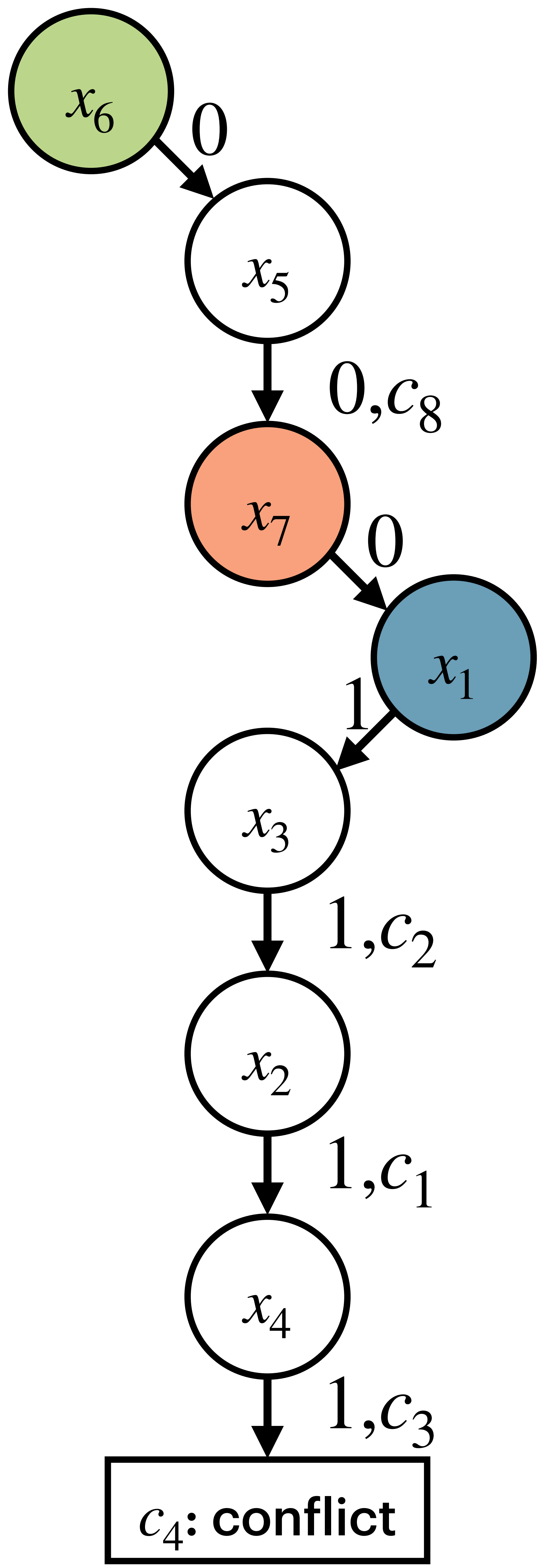
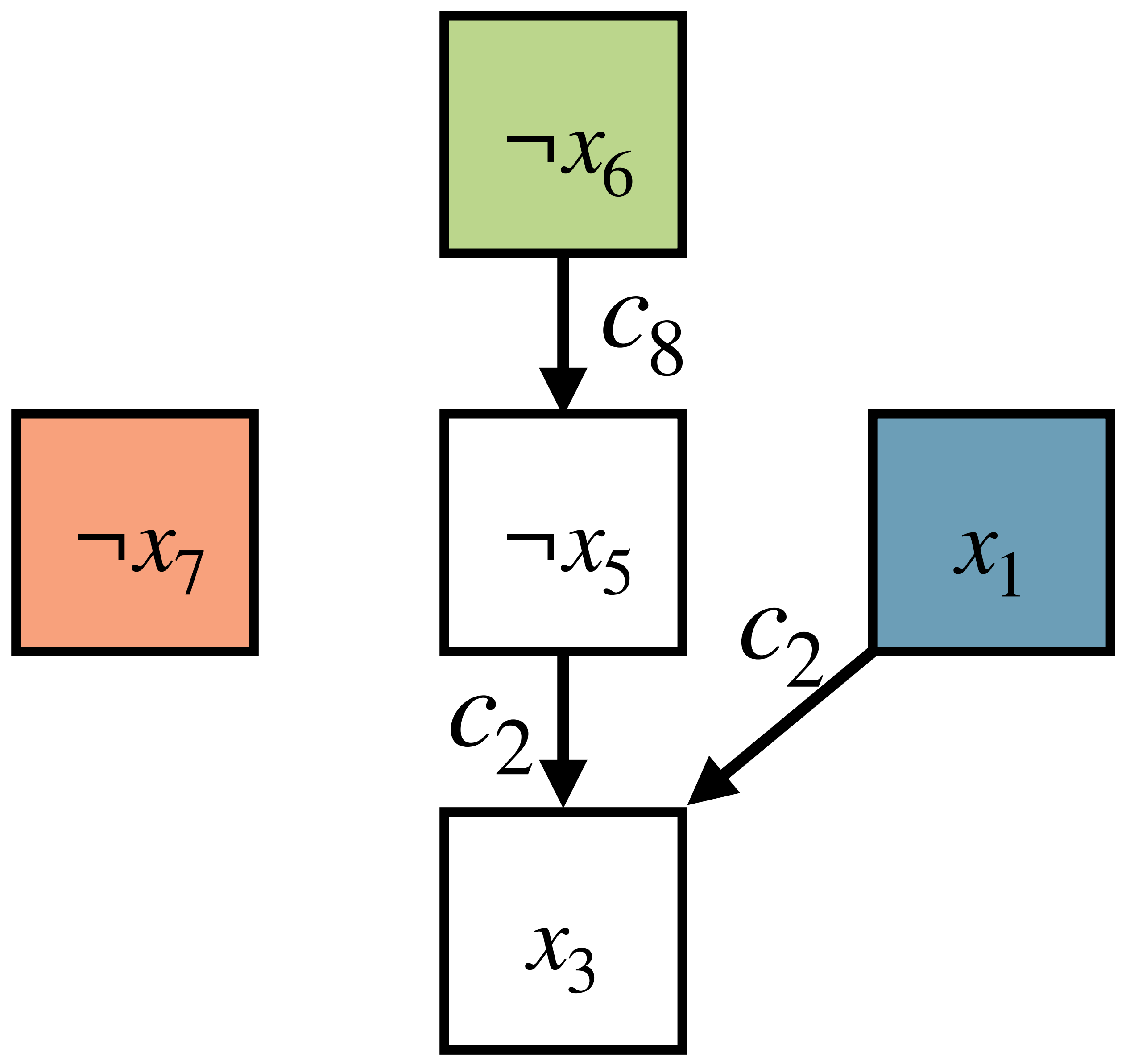
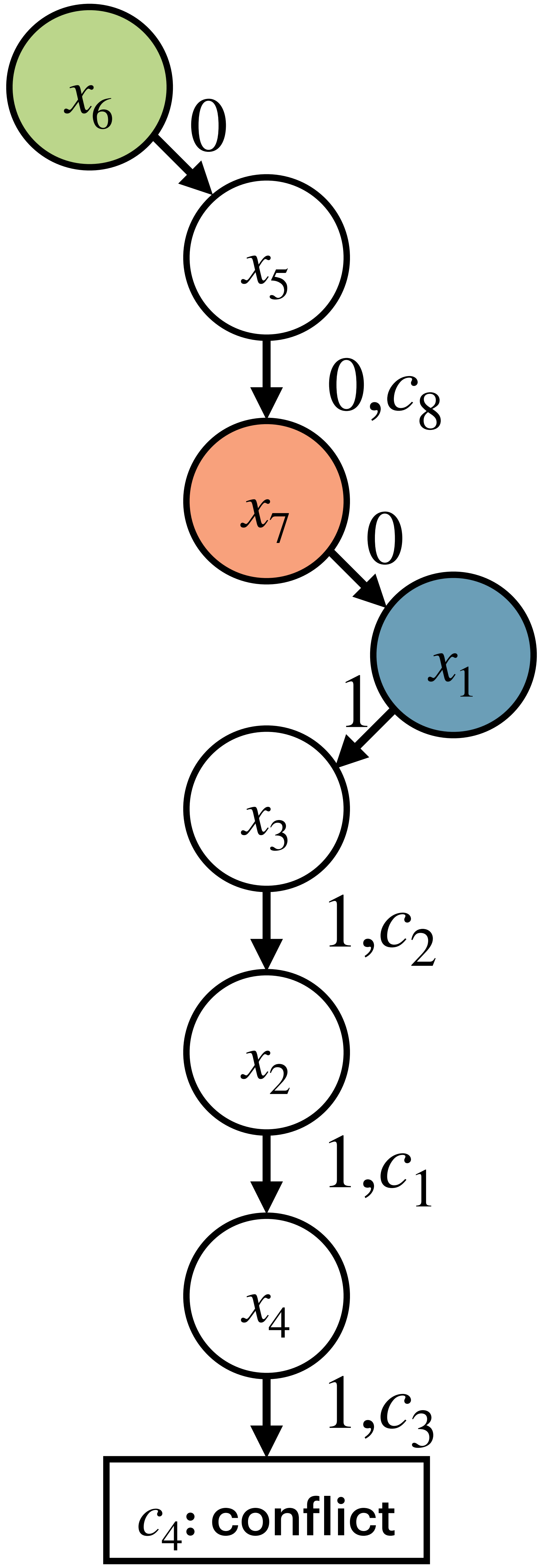
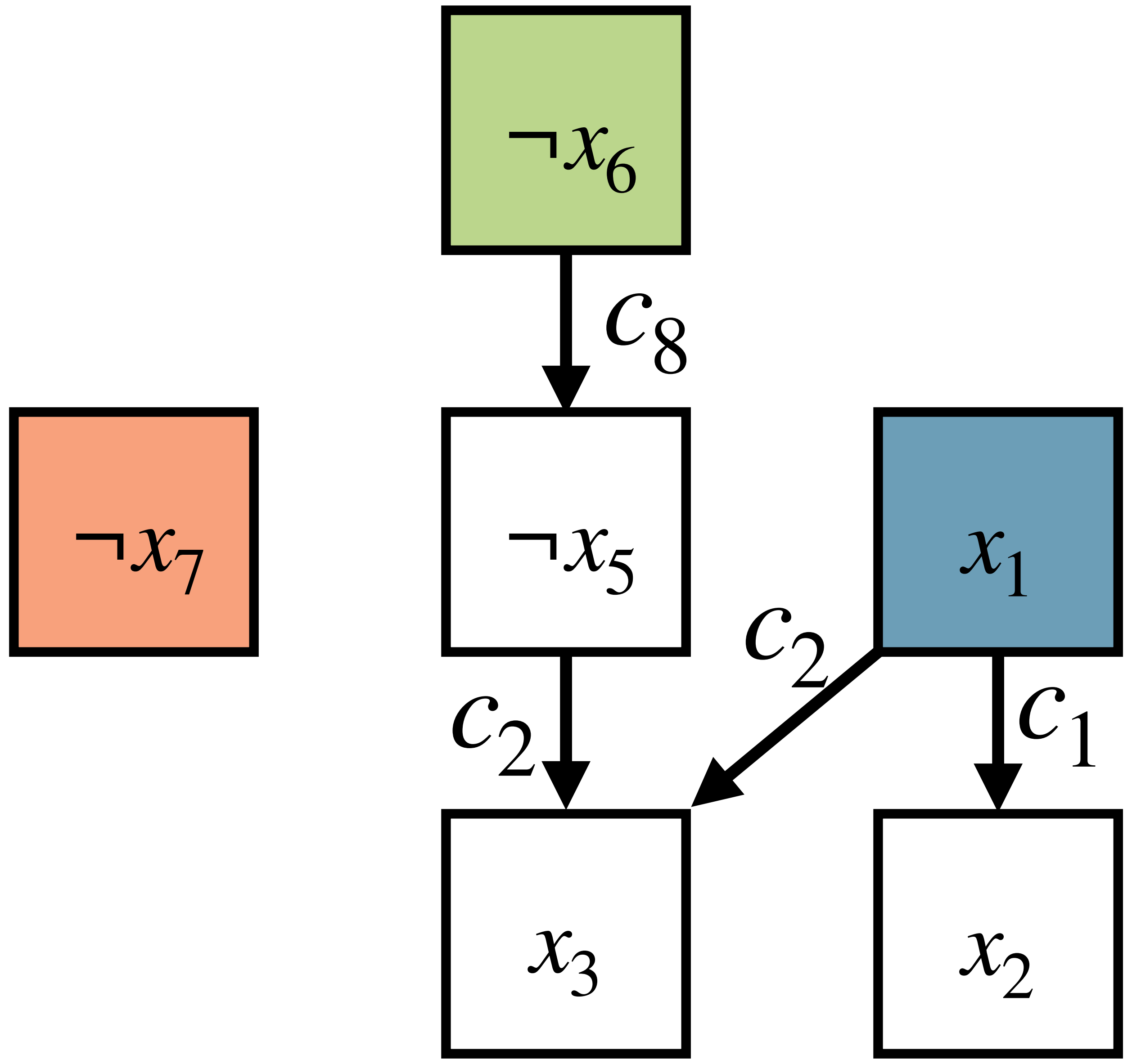
$$c_5 = (x_1 \lor x_5 \lor \neg x_2)$$
$$c_6 = (x_2 \lor x_3)$$
$$c_7 = (x_2 \lor \neg x_3 \lor x_7)$$
$$c_8 = (x_6 \lor \neg x_5)$$





$$c_1 = (\neg x_1 \lor x_2)$$
$$c_2 = (\neg x_1 \lor x_3 \lor x_5)$$
$$c_3 = (\neg x_2 \lor x_4)$$
$$c_4 = (\neg x_3 \lor \neg x_4)$$
$$c_5 = (x_1 \lor x_5 \lor \neg x_2)$$
$$c_6 = (x_2 \lor x_3)$$
$$c_7 = (x_2 \lor \neg x_3 \lor x_7)$$
$$c_8 = (x_6 \lor \neg x_5)$$





$$c_1 = (\neg x_1 \lor x_2)$$

$$c_2 = (\neg x_1 \lor x_3 \lor x_5)$$

$$c_3 = (\neg x_2 \lor x_4)$$

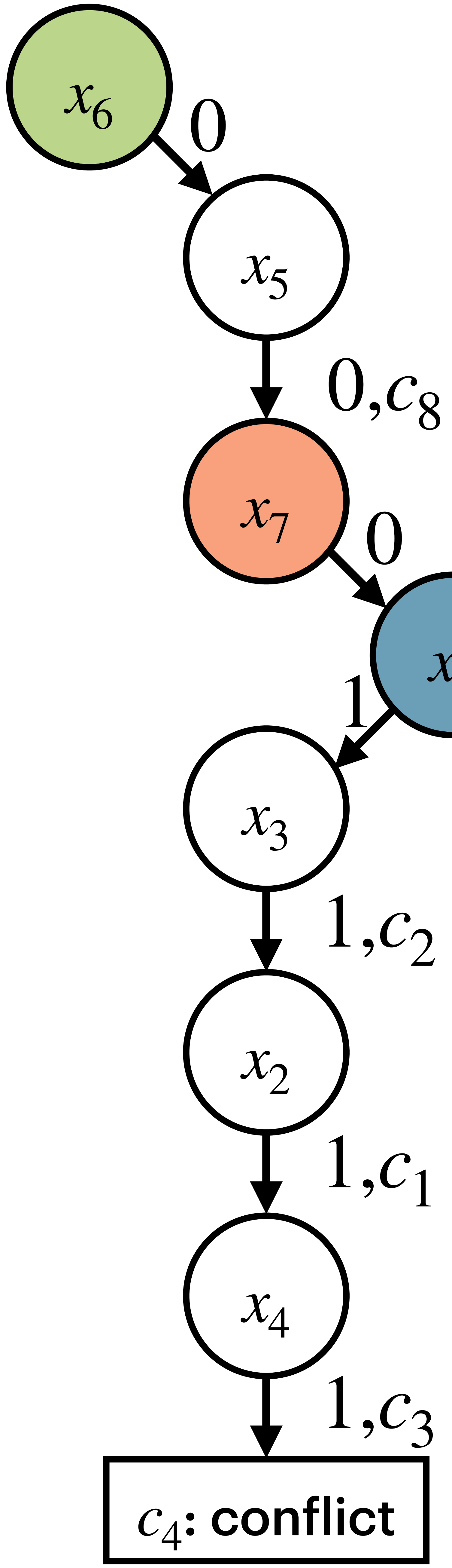
$$c_4 = (\neg x_3 \lor \neg x_4)$$

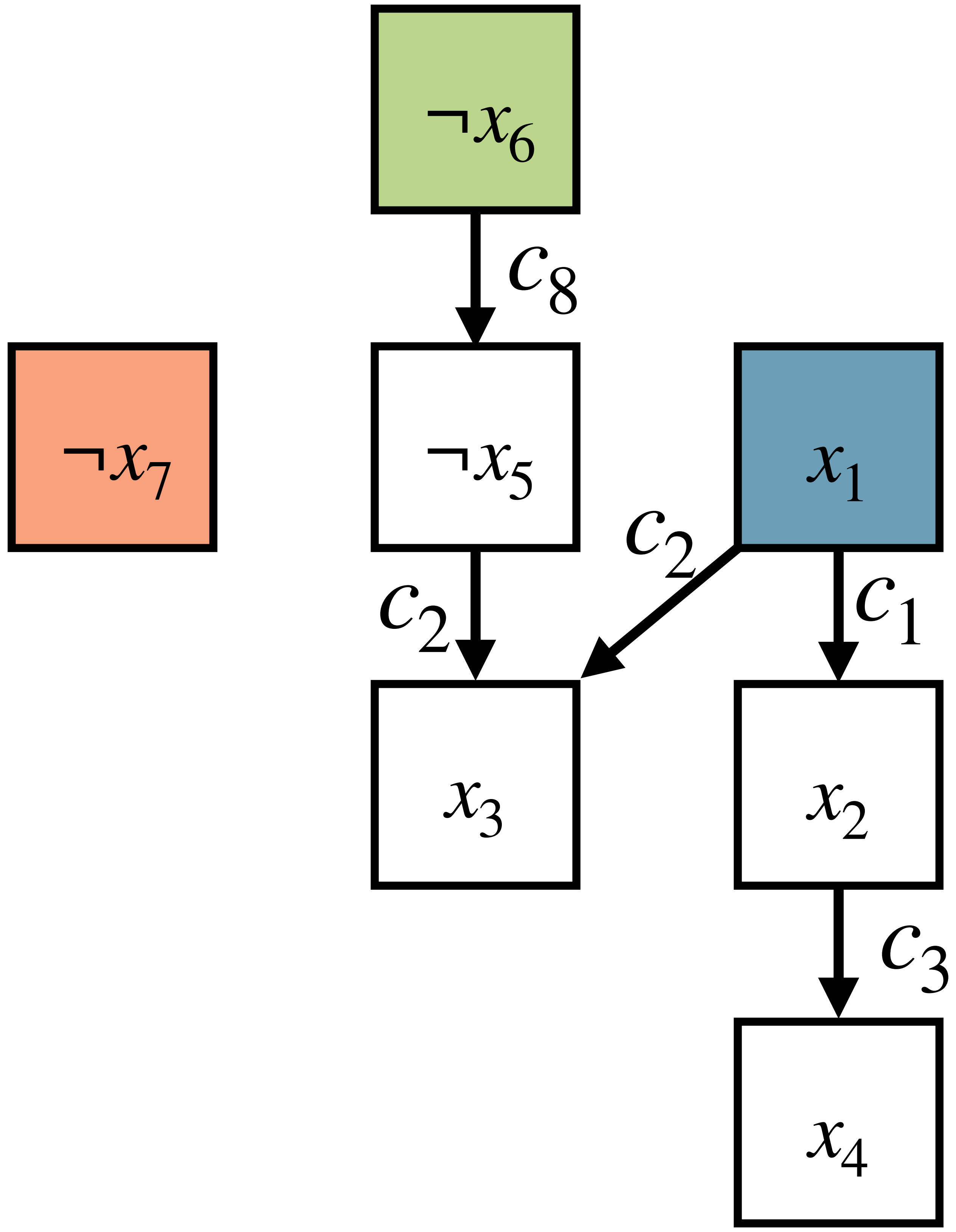
$$c_5 = (x_1 \lor x_5 \lor \neg x_2)$$

$$c_6 = (x_2 \lor x_3)$$

$$c_7 = (x_2 \lor \neg x_3 \lor x_7)$$

$$c_8 = (x_6 \lor \neg x_5)$$





$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_5)$$

$$c_3 = (\neg x_2 \vee x_4)$$

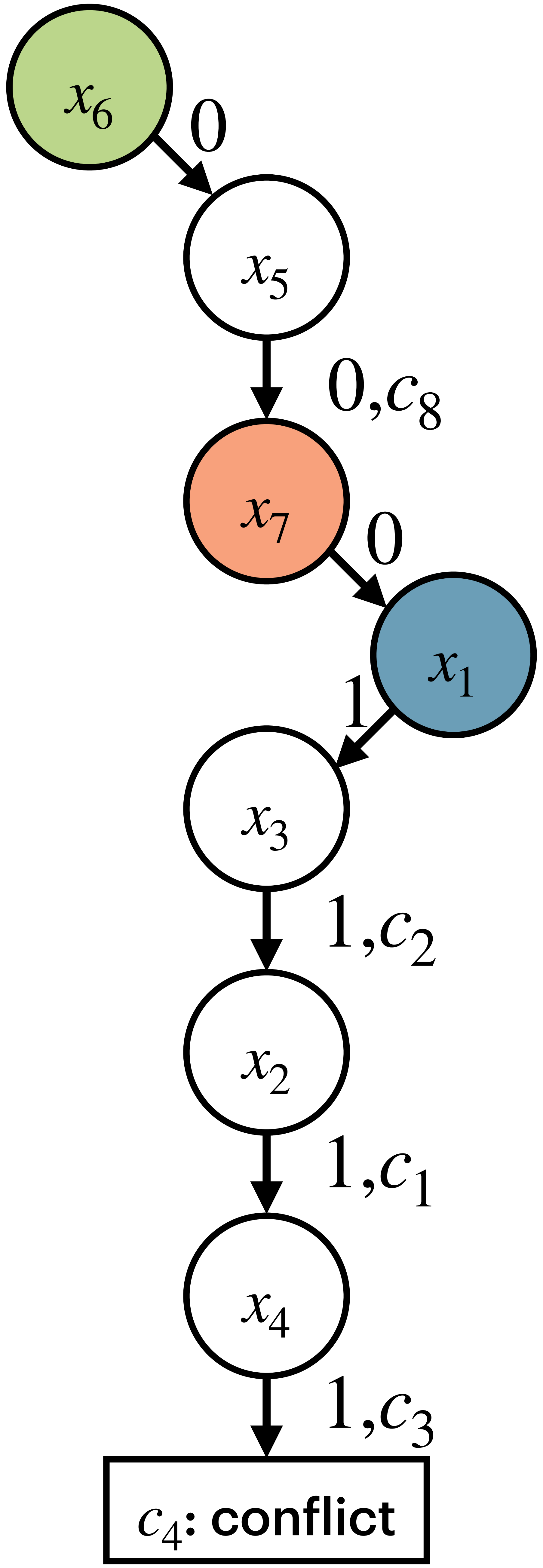
$$c_4 = (\neg x_3 \vee \neg x_4)$$

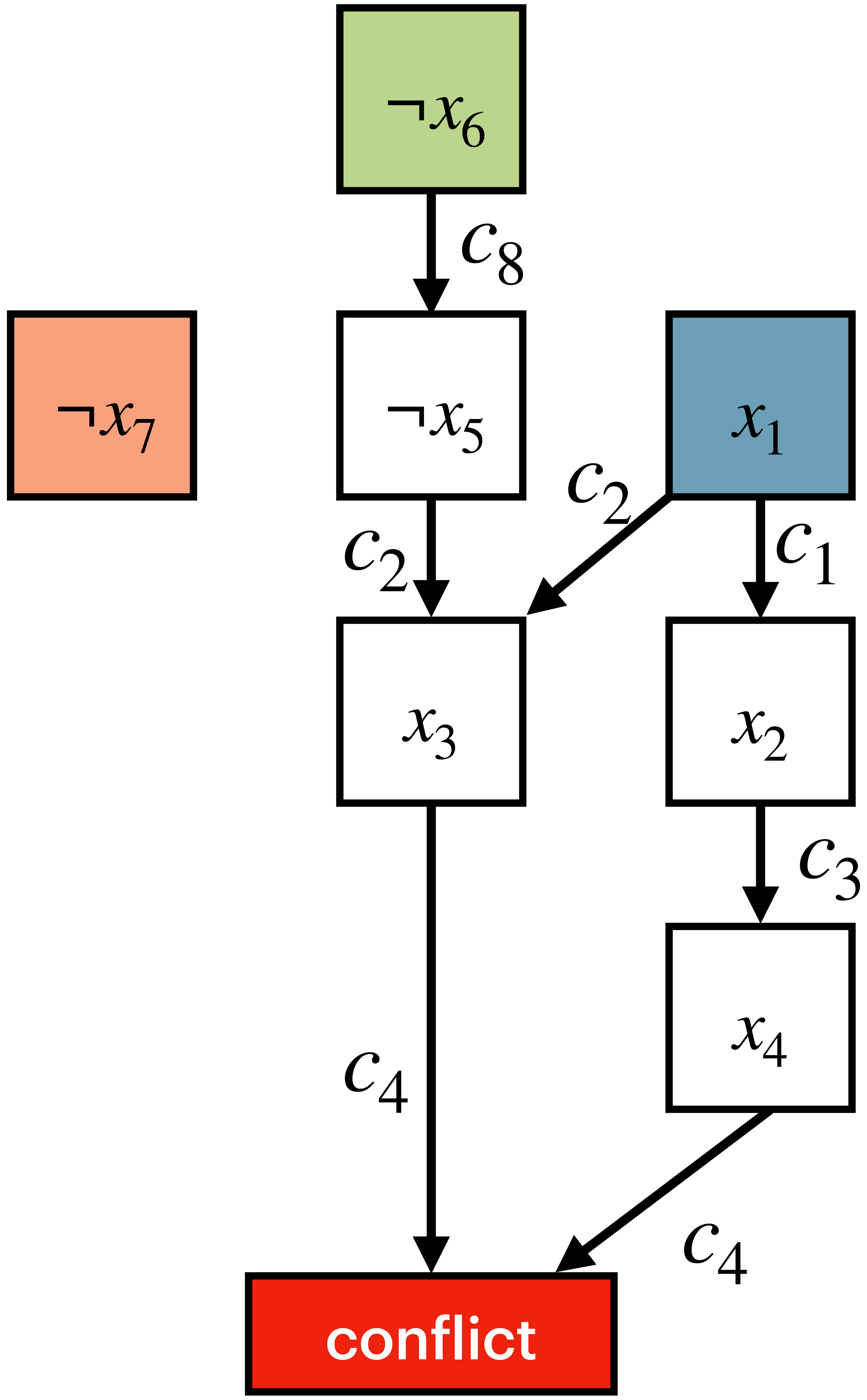
$$c_5 = (x_1 \vee x_5 \vee \neg x_2)$$

$$c_6 = (x_2 \vee x_3)$$

$$c_7 = (x_2 \vee \neg x_3 \vee x_7)$$

$$c_8 = (x_6 \vee \neg x_5)$$





$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_5)$$

$$c_3 = (\neg x_2 \vee x_4)$$

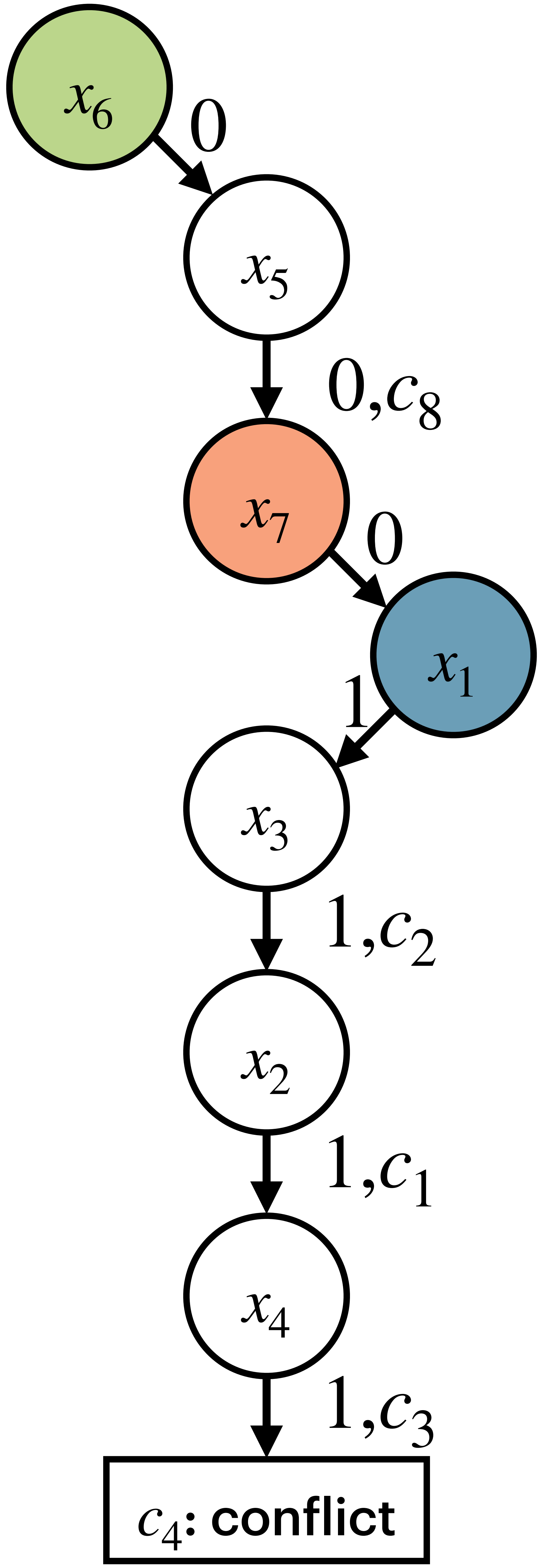
$$c_4 = (\neg x_3 \vee \neg x_4)$$

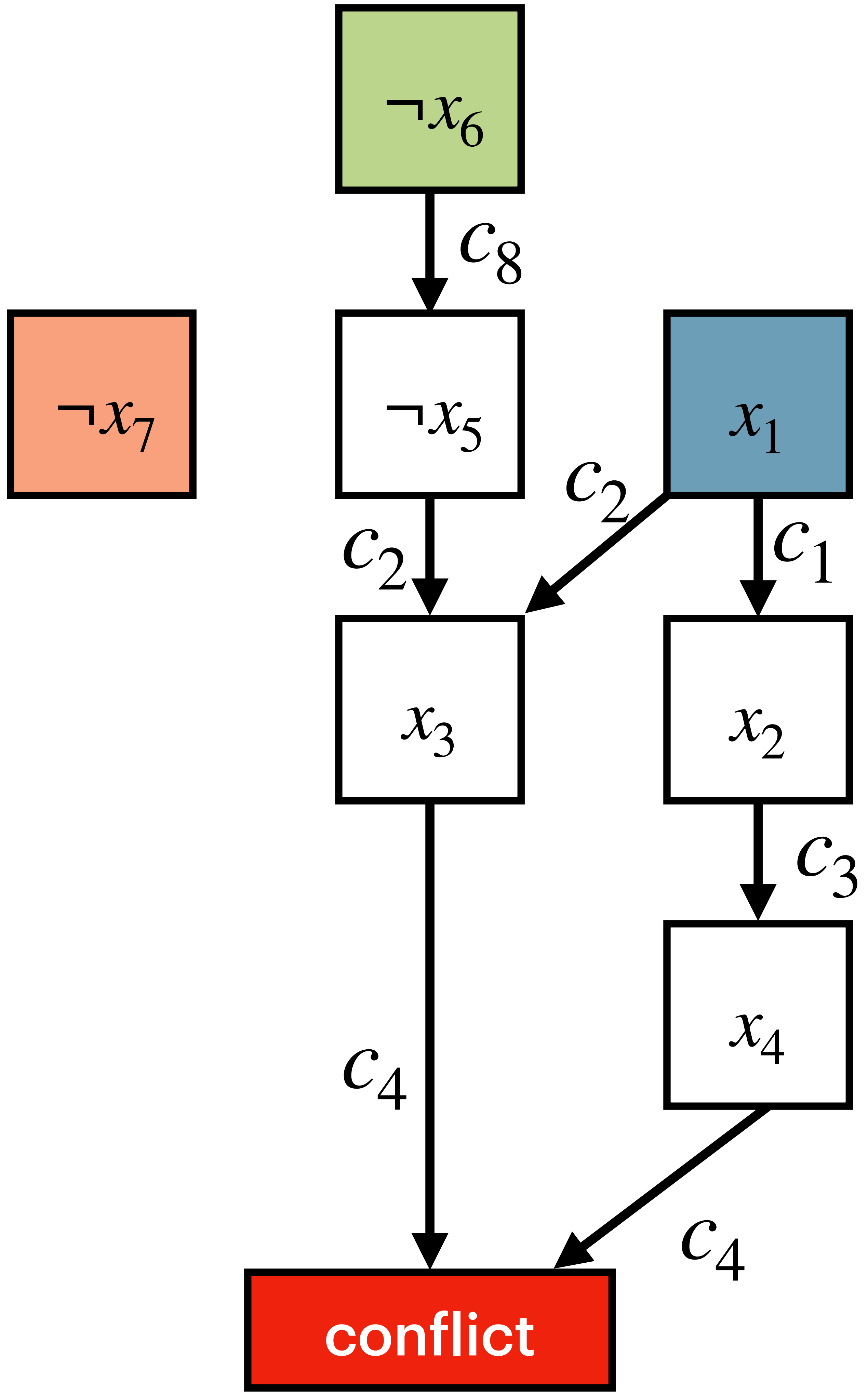
$$c_5 = (x_1 \vee x_5 \vee \neg x_2)$$

$$c_6 = (x_2 \vee x_3)$$

$$c_7 = (x_2 \vee \neg x_3 \vee x_7)$$

$$c_8 = (x_6 \vee \neg x_5)$$





$$\neg x_6 \wedge x_1 \rightarrow \bot$$

$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_5)$$

$$c_3 = (\neg x_2 \vee x_4)$$

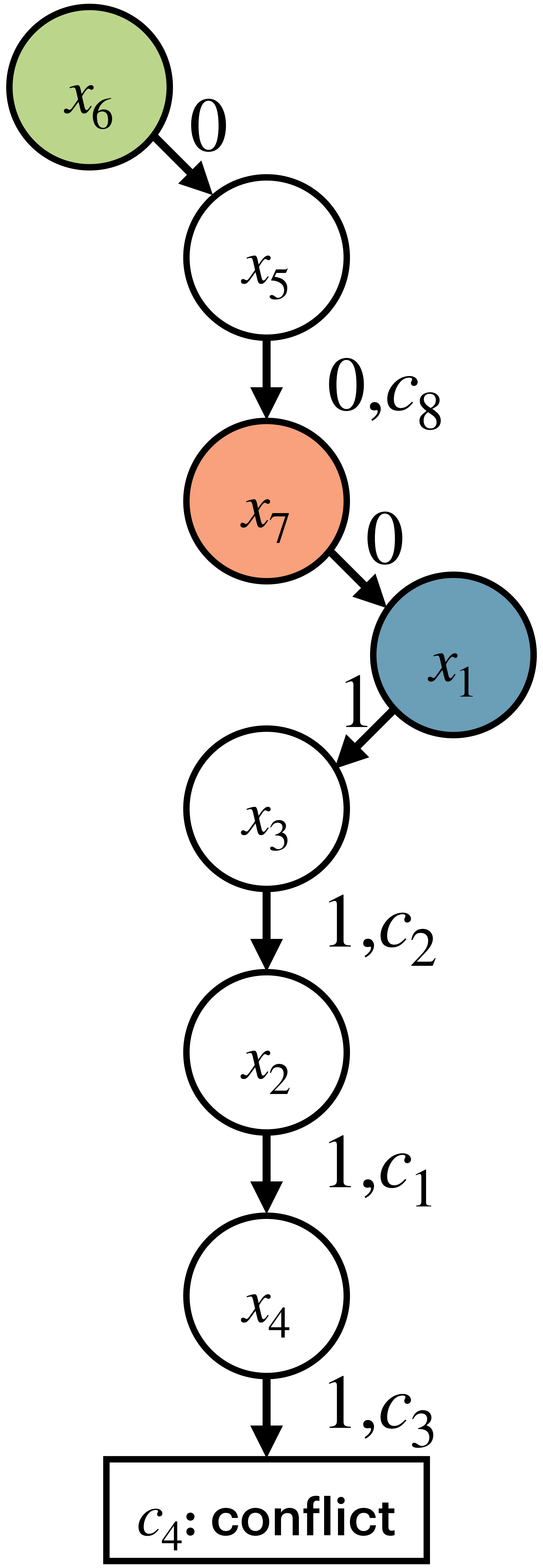
$$c_4 = (\neg x_3 \vee \neg x_4)$$

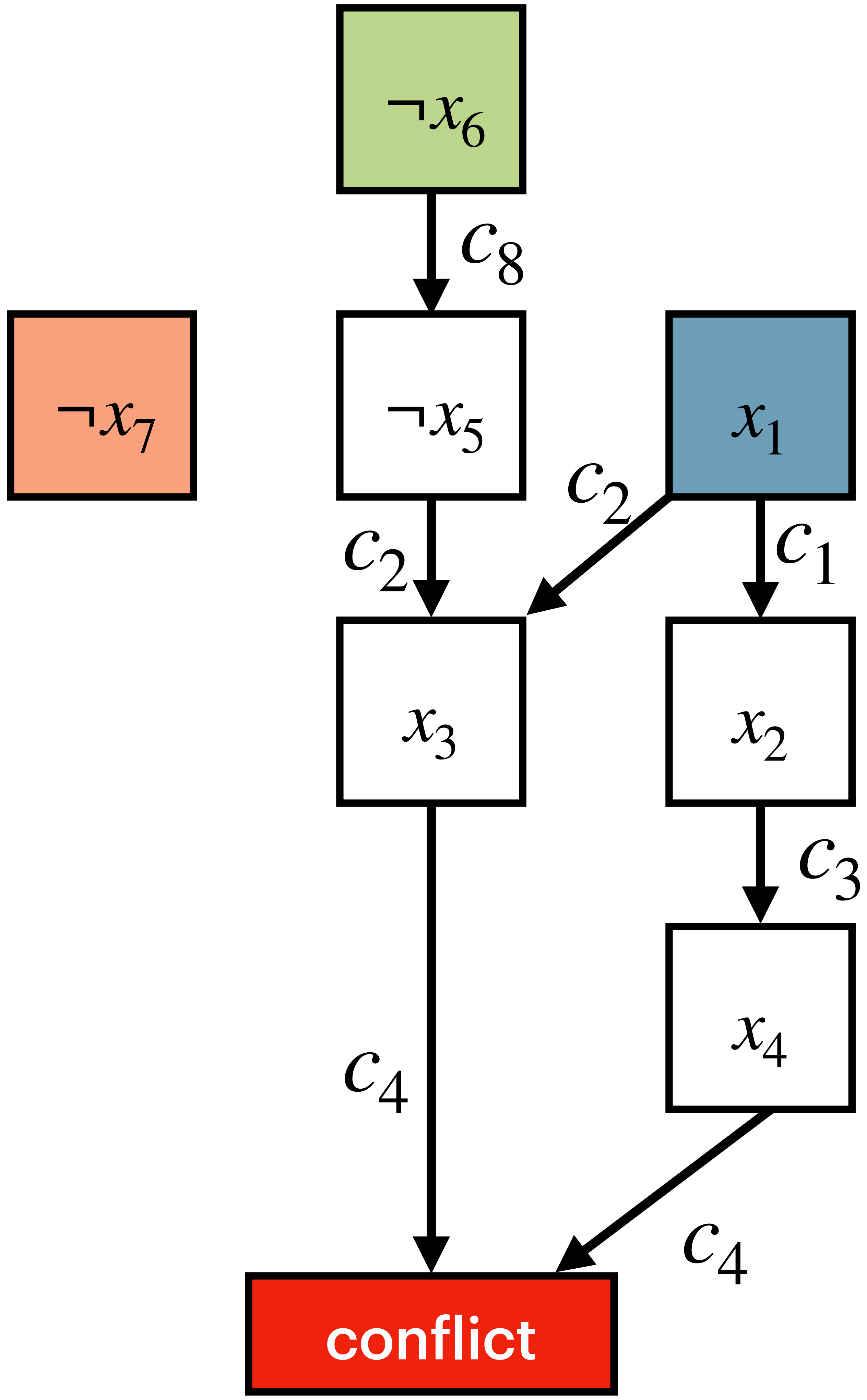
$$c_5 = (x_1 \vee x_5 \vee \neg x_2)$$

$$c_6 = (x_2 \vee x_3)$$

$$c_7 = (x_2 \vee \neg x_3 \vee x_7)$$

$$c_8 = (x_6 \vee \neg x_5)$$





$$x_6 \vee \neg x_1$$

$$c_1 = (\neg x_1 \lor x_2)$$

$$c_2 = (\neg x_1 \lor x_3 \lor x_5)$$

$$c_3 = (\neg x_2 \lor x_4)$$

$$c_4 = (\neg x_3 \lor \neg x_4)$$

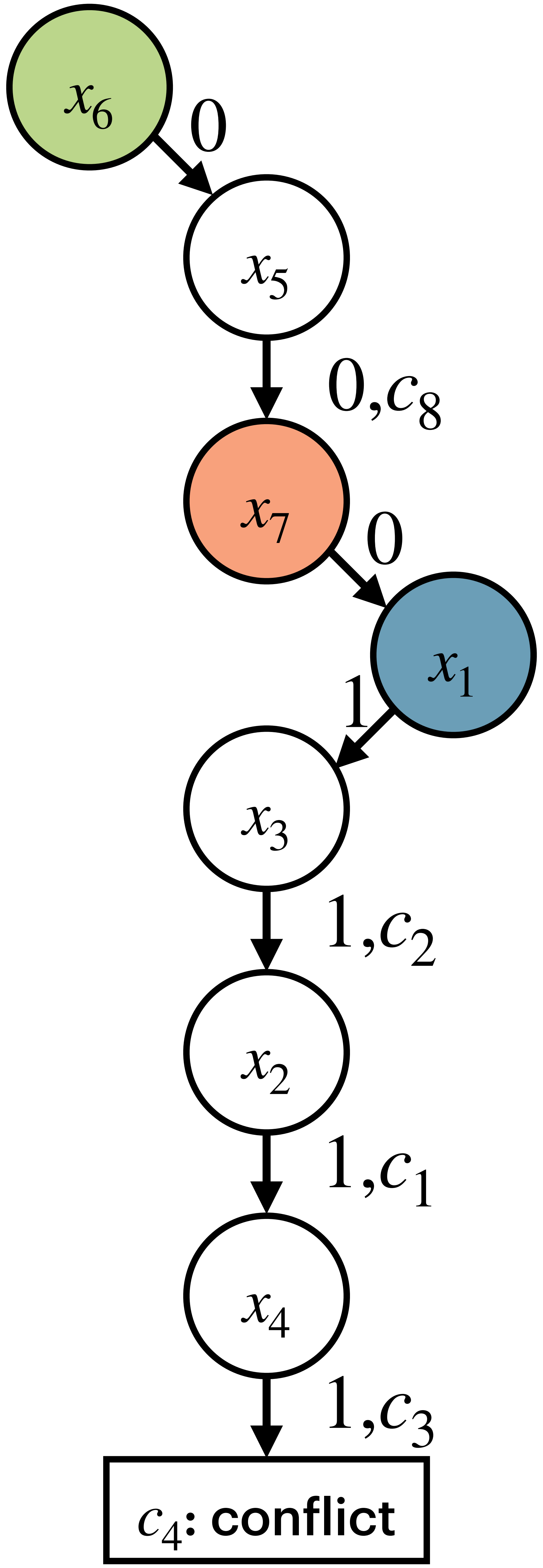
$$c_5 = (x_1 \lor x_5 \lor \neg x_2)$$

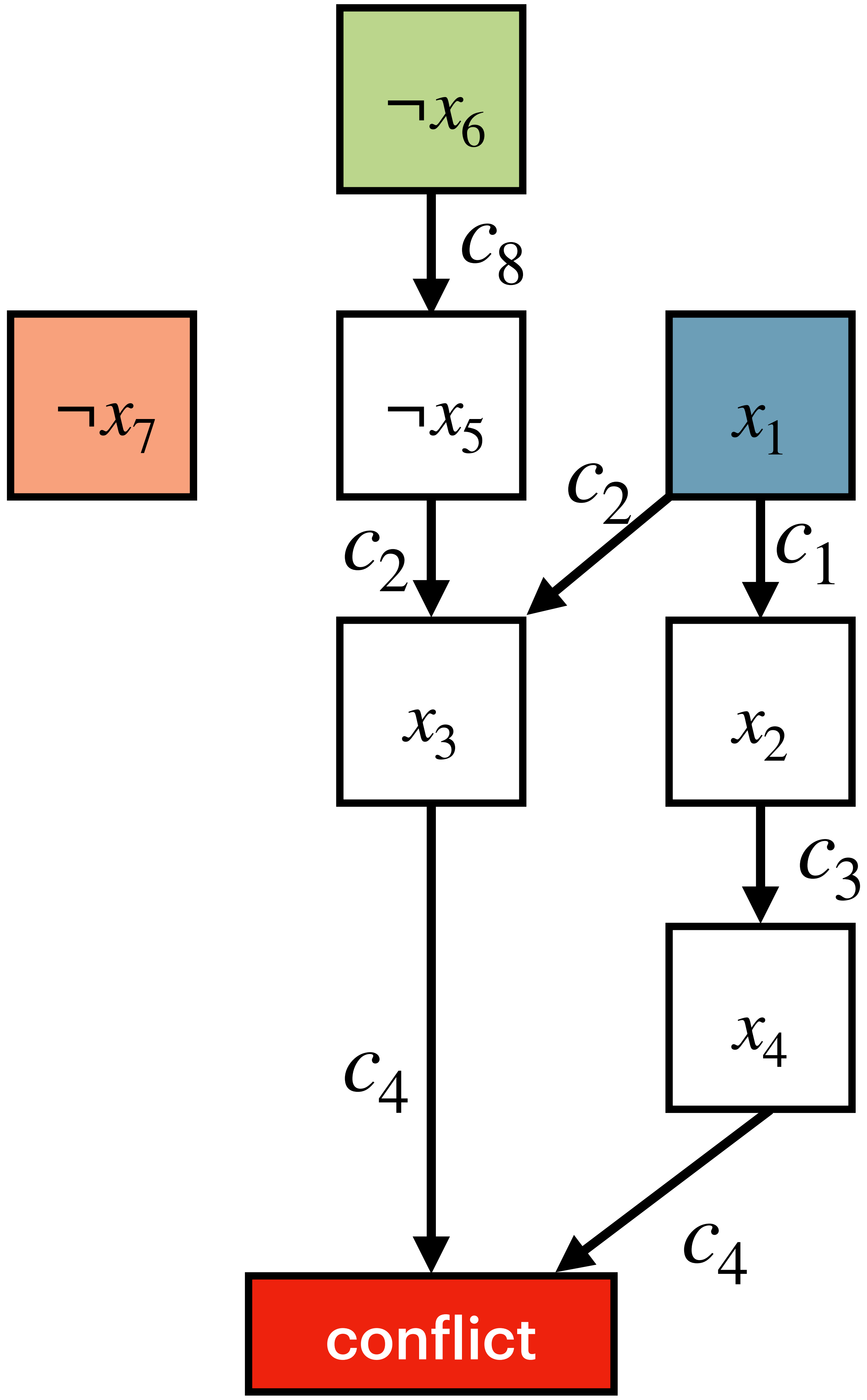
$$c_6 = (x_2 \lor x_3)$$

$$c_7 = (x_2 \lor \neg x_3 \lor x_7)$$

$$c_8 = (x_6 \lor \neg x_5)$$

$$c_9 = x_6 \lor \neg x_1$$





$c_1 = (\neg x_1 \lor x_2)$

$c_2 = (\neg x_1 \lor x_3 \lor x_5)$

$c_3 = (\neg x_2 \lor x_4)$

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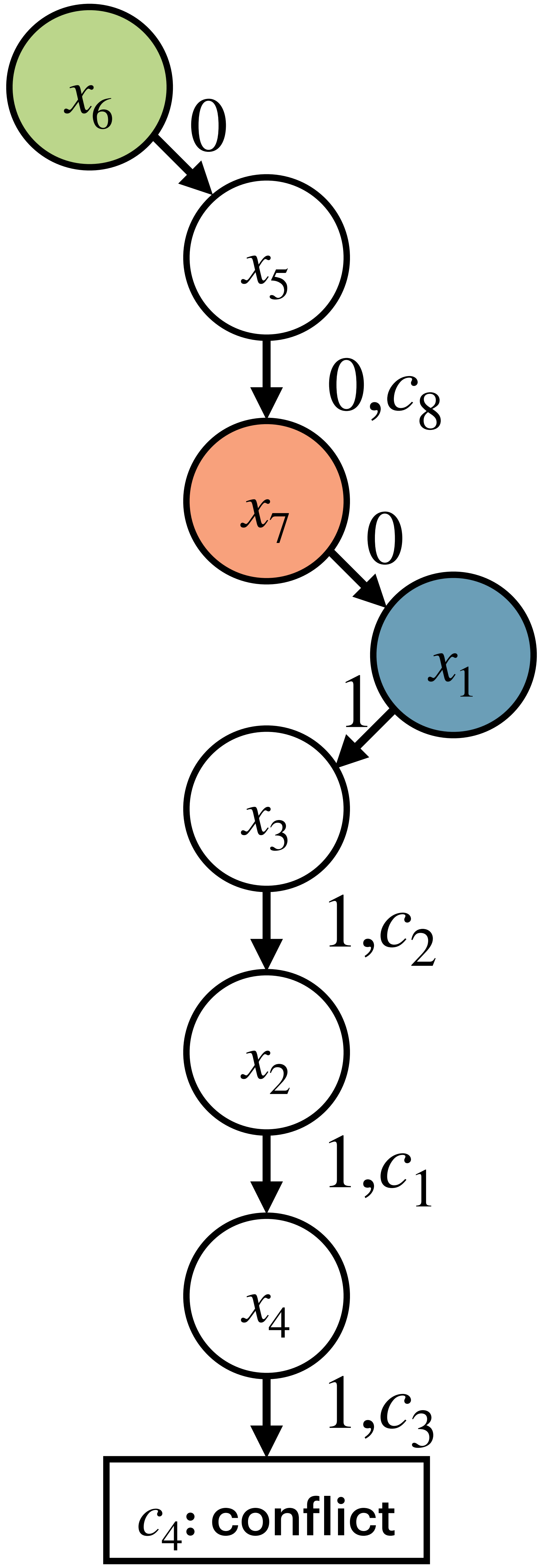
$c_6 = (x_2 \lor x_3)$

$c_7 = (x_2 \lor \neg x_3 \lor x_7)$

$c_8 = (x_6 \lor \neg x_5)$

$c_9 = x_6 \lor \neg x_1$



$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_5)$$

$$c_3 = (\neg x_2 \vee x_4)$$

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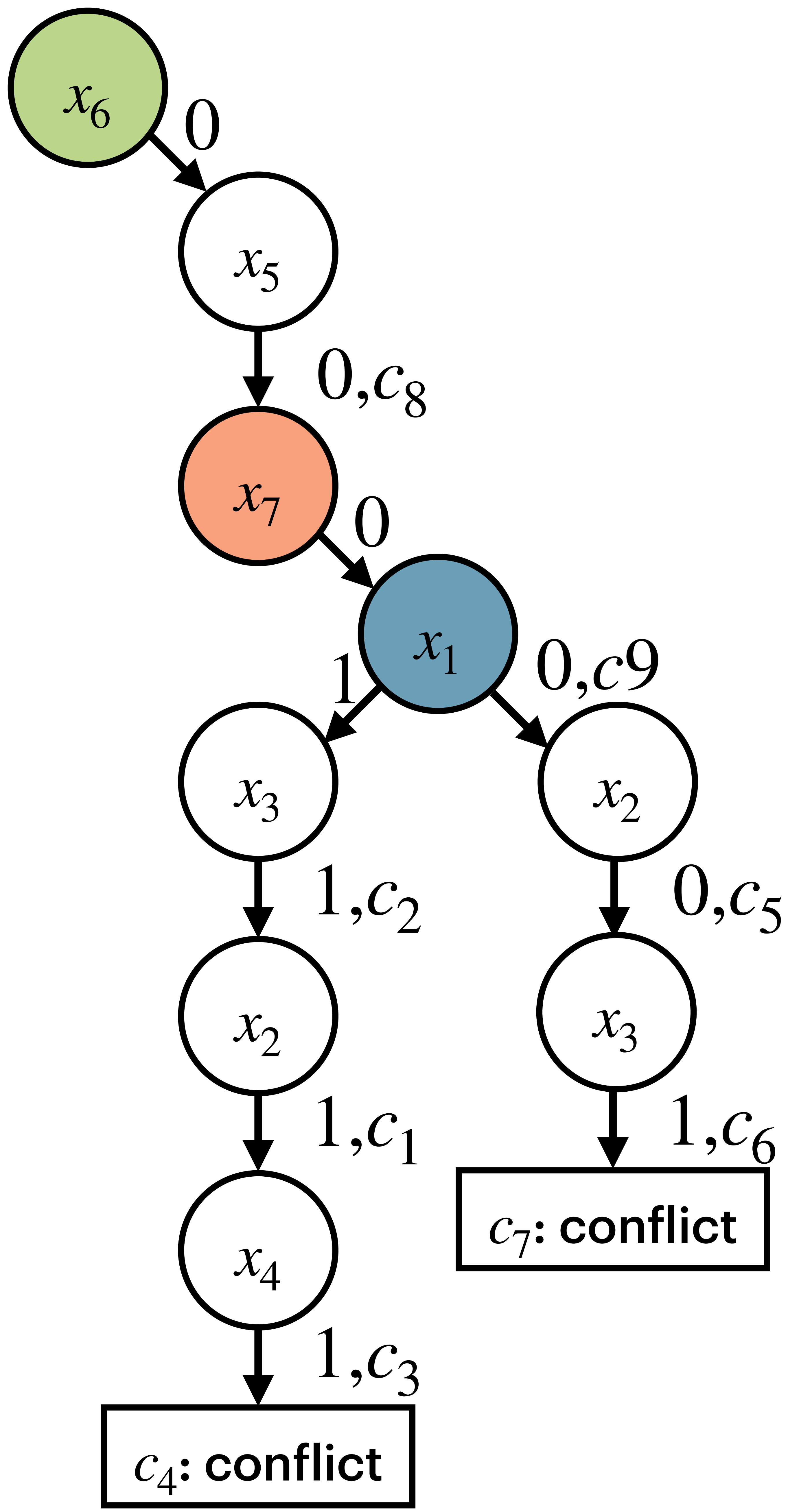
$$c_9 = x_6 \vee \neg x_1$$



$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_5)$$

$$c_3 = (\neg x_2 \vee x_4)$$

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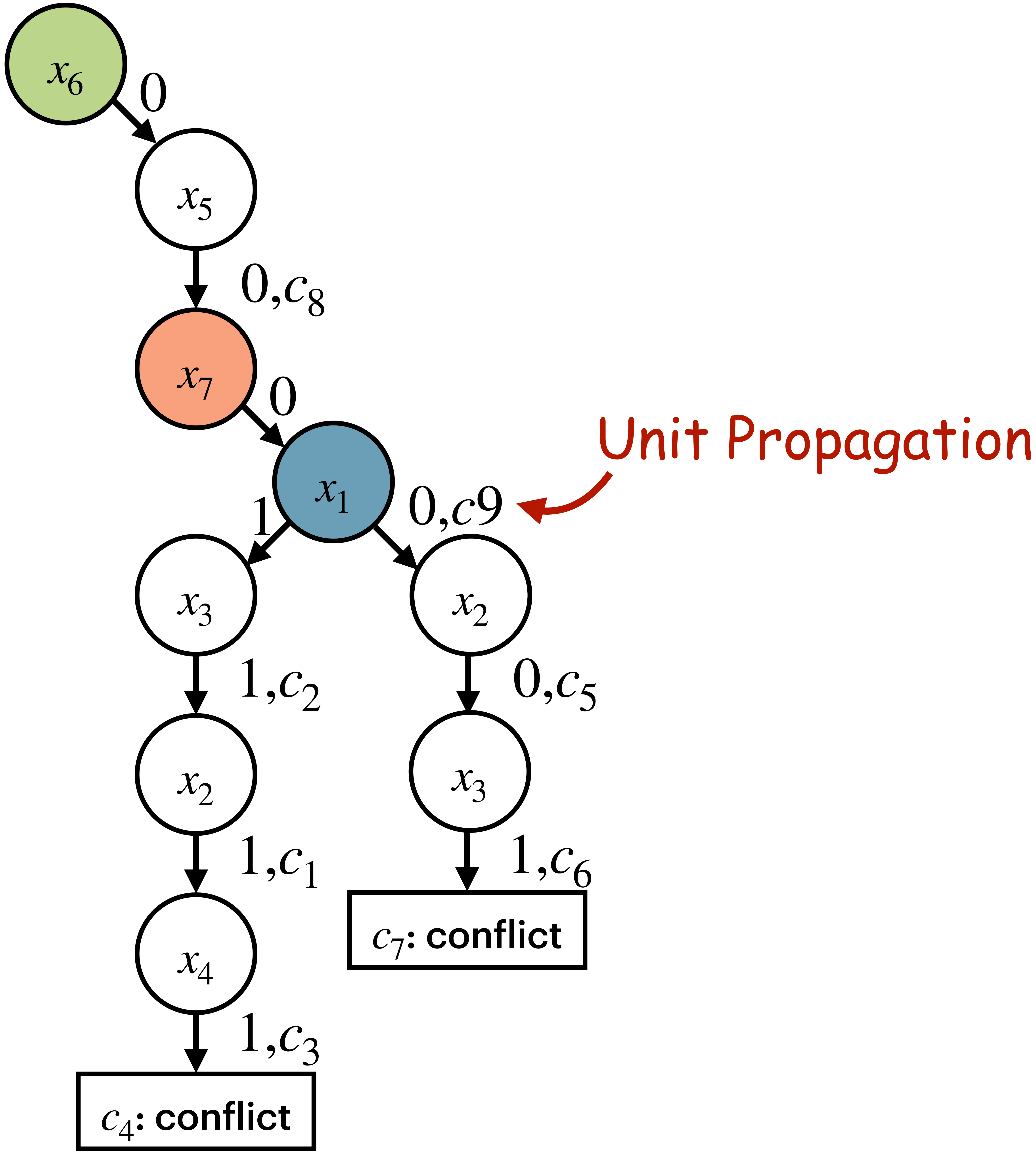

$$c_1 = (\neg x_1 \vee x_2)$$

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$$c_7 = (x_2 \vee \neg x_3 \vee x_7)$$

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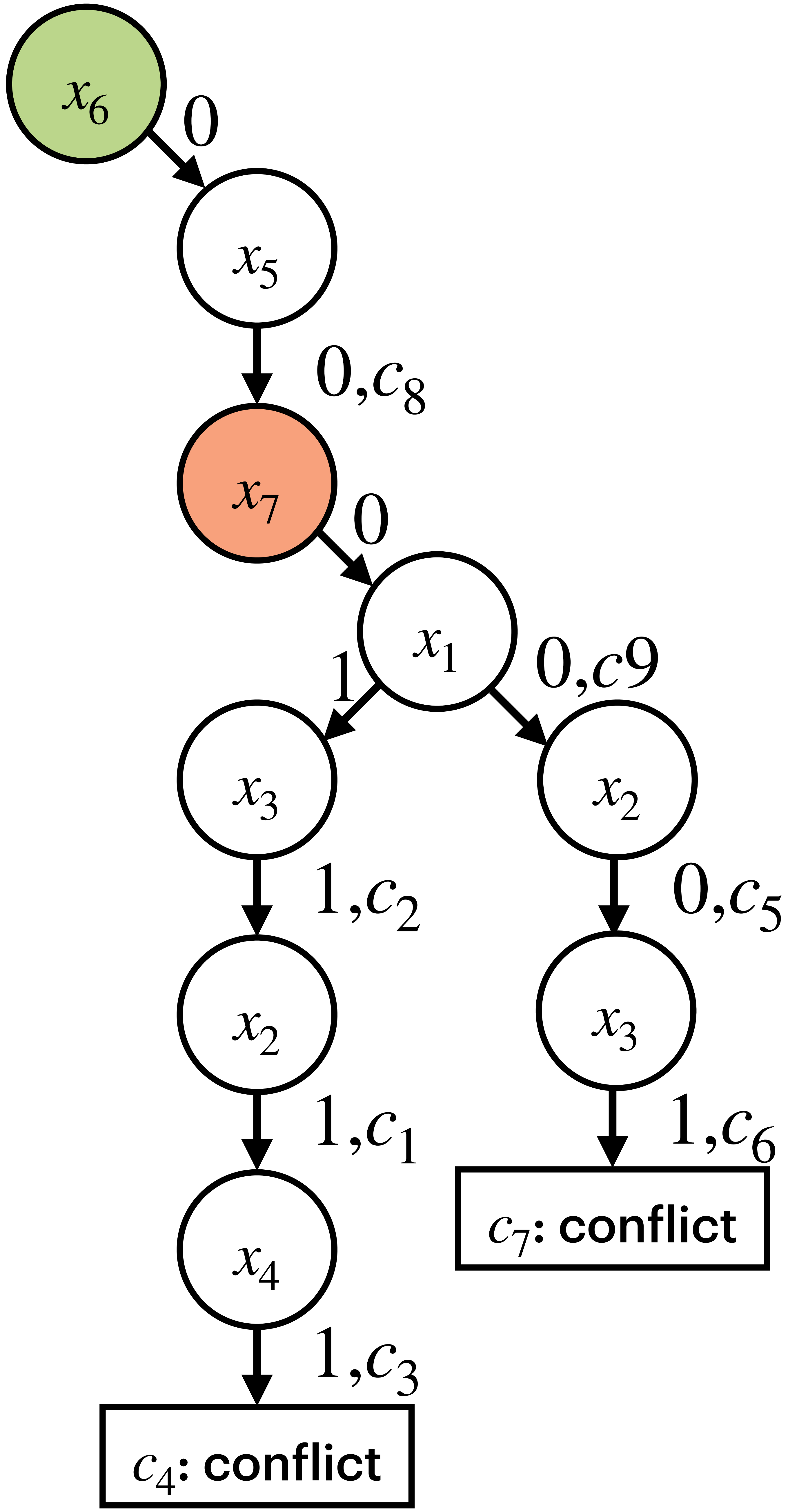

$$c_1 = (\neg x_1 \vee x_2)$$

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$$c_7 = (x_2 \vee \neg x_3 \vee x_7)$$

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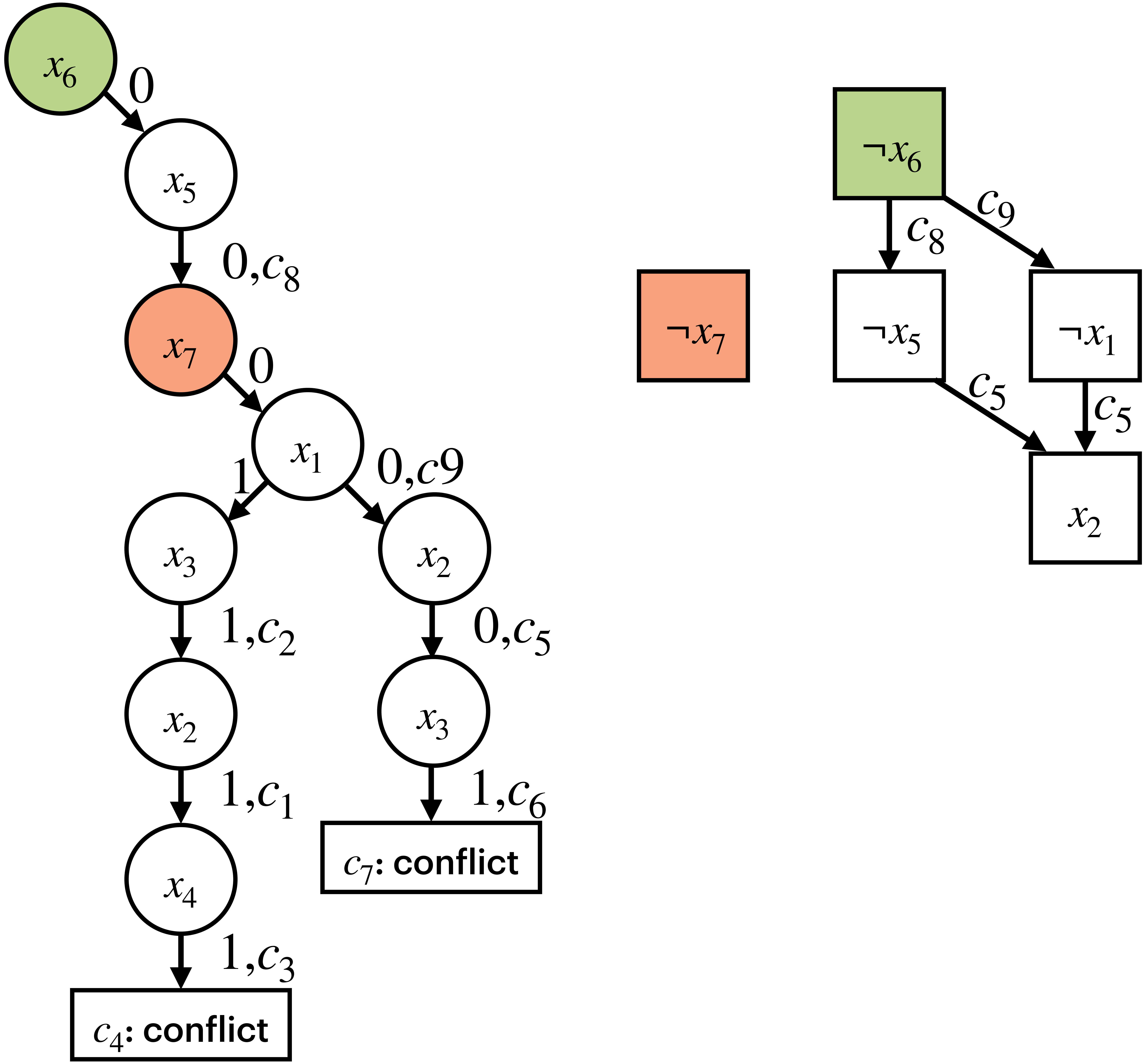

$$c_1 = (\neg x_1 \vee x_2)$$

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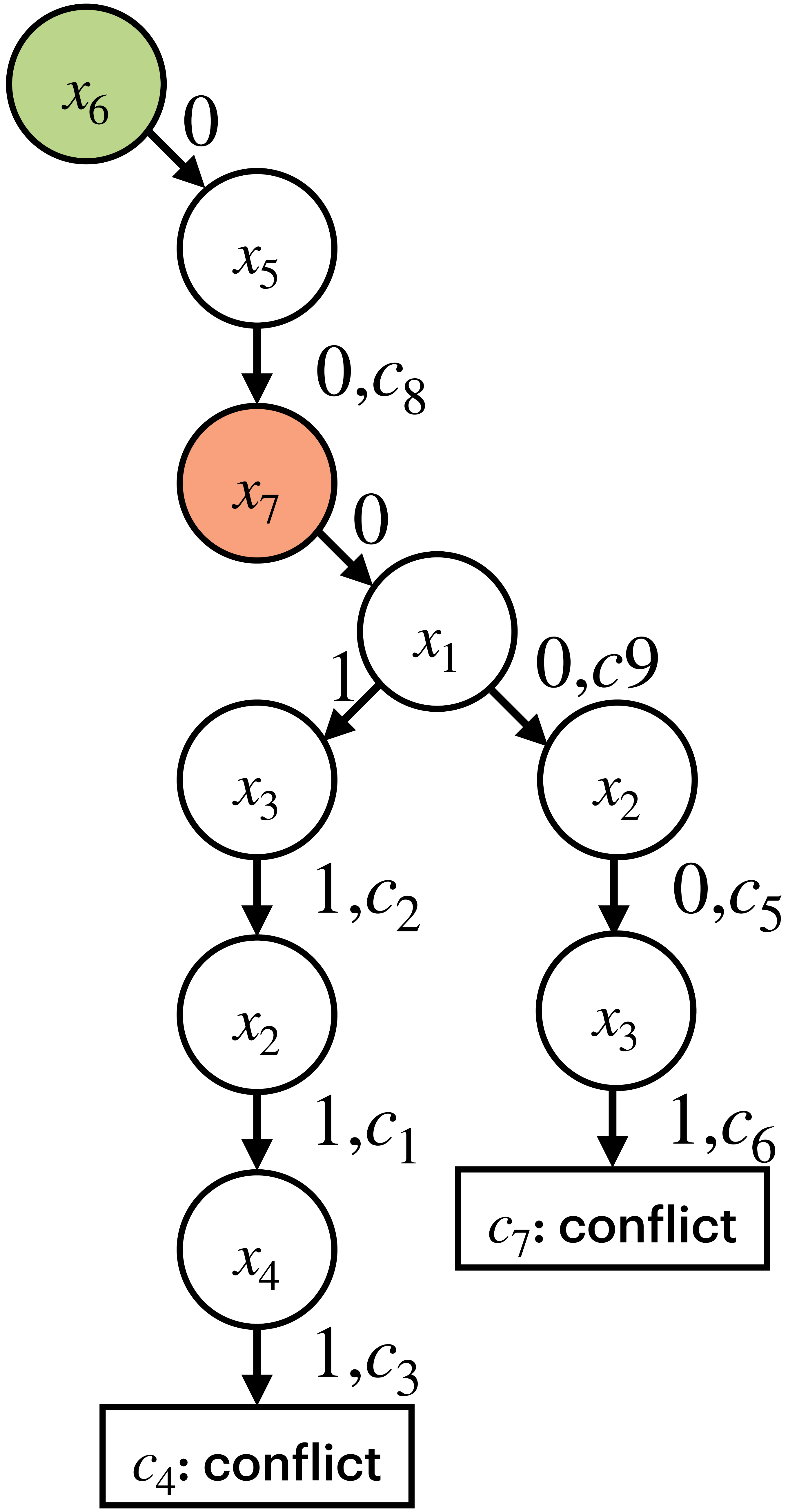

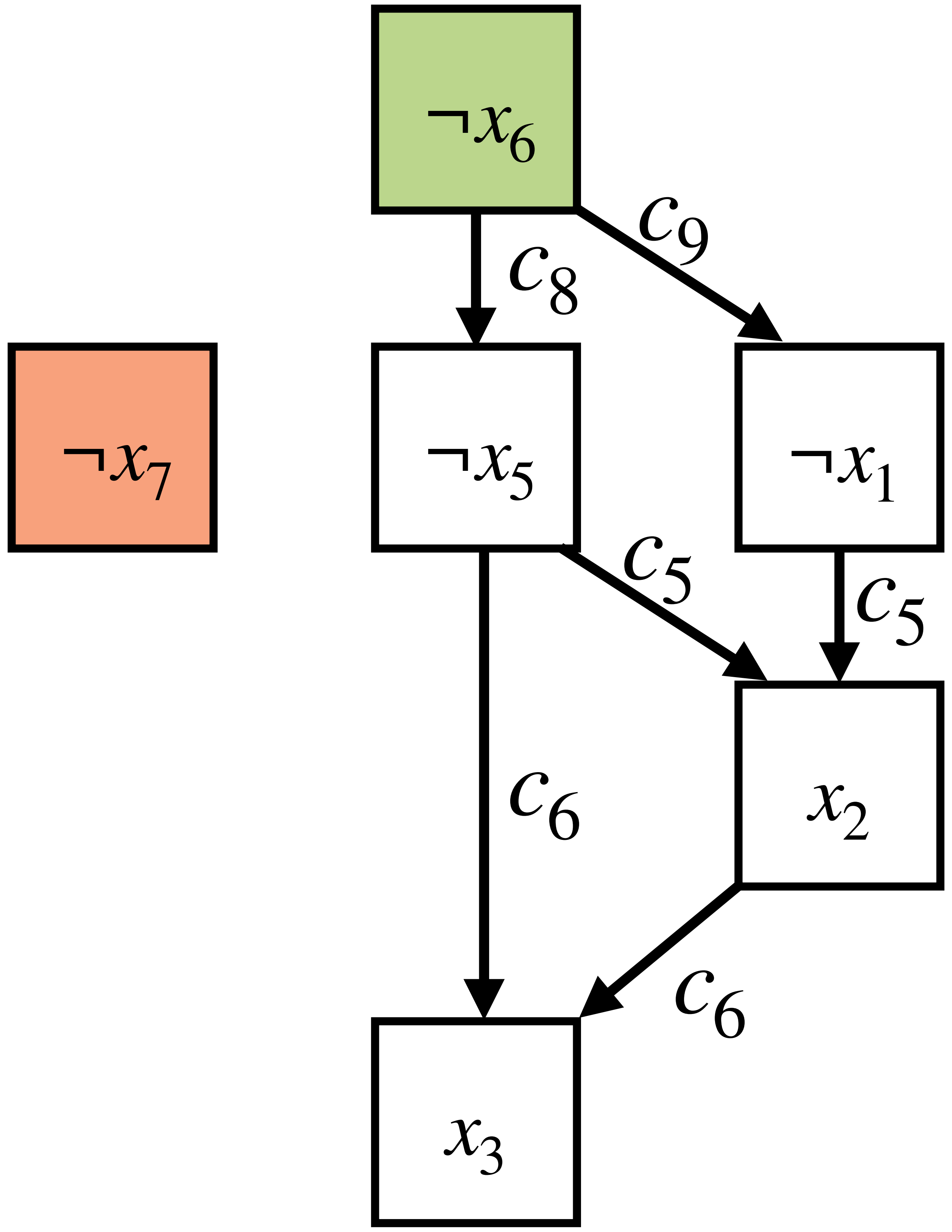

$$c_1 = (\neg x_1 \lor x_2)$$

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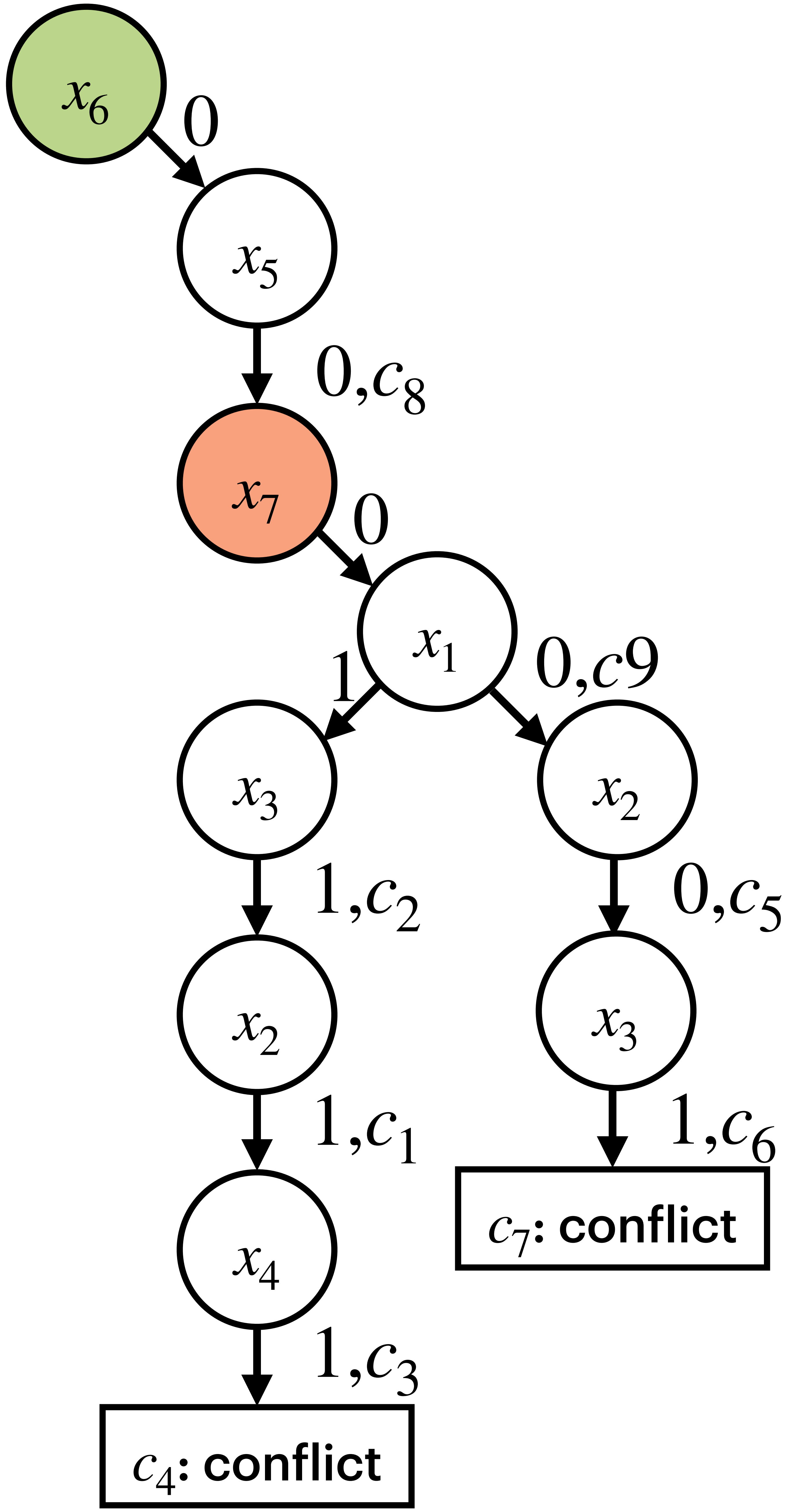

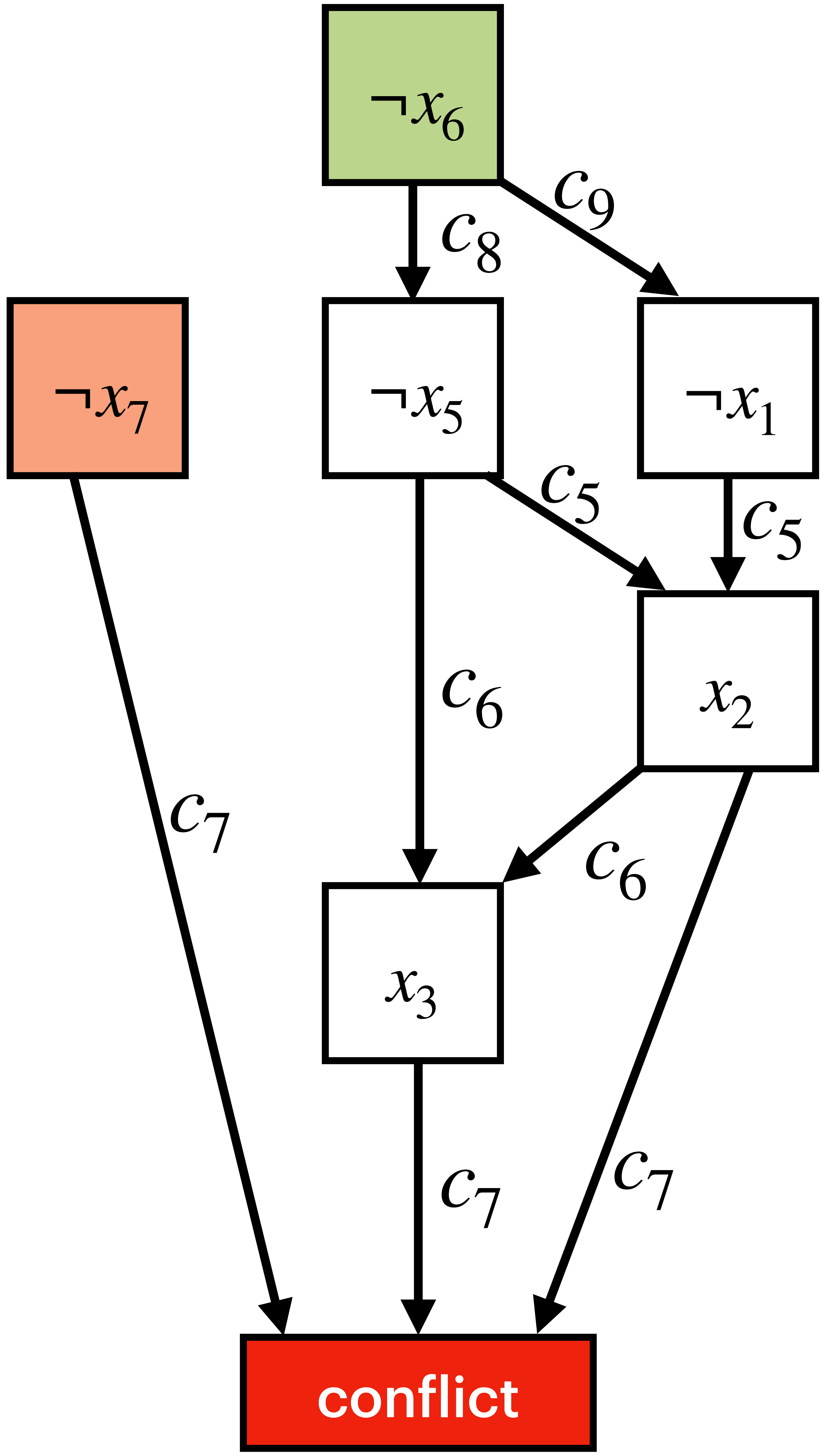

$$c_1 = (\neg x_1 \lor x_2)$$

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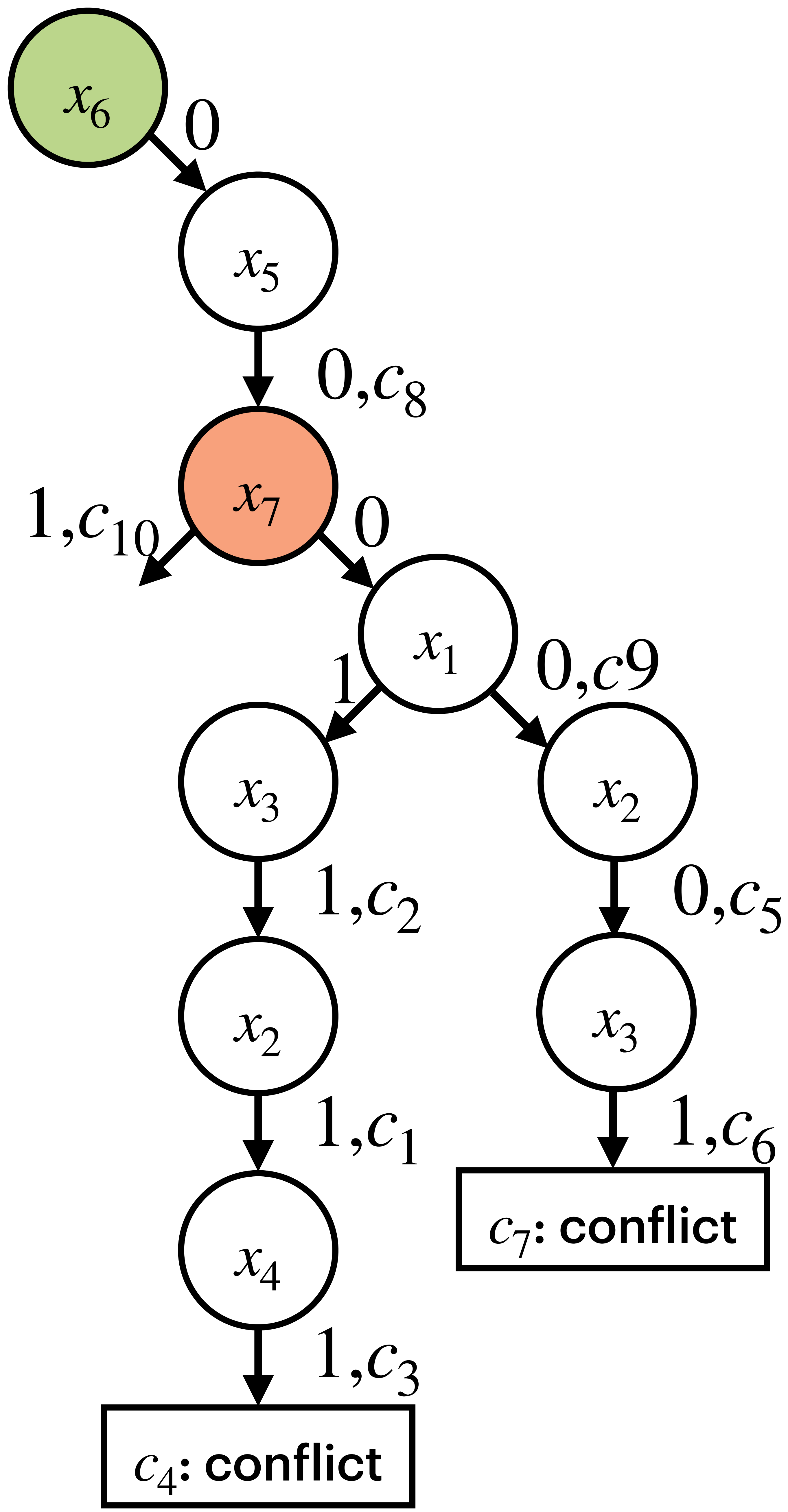
$$c_8 = (x_6 \lor \neg x_5)$$

$$c_9 = x_6 \lor \neg x_1$$

$$c_{10} = x_7 \lor x_6$$



$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_5)$$

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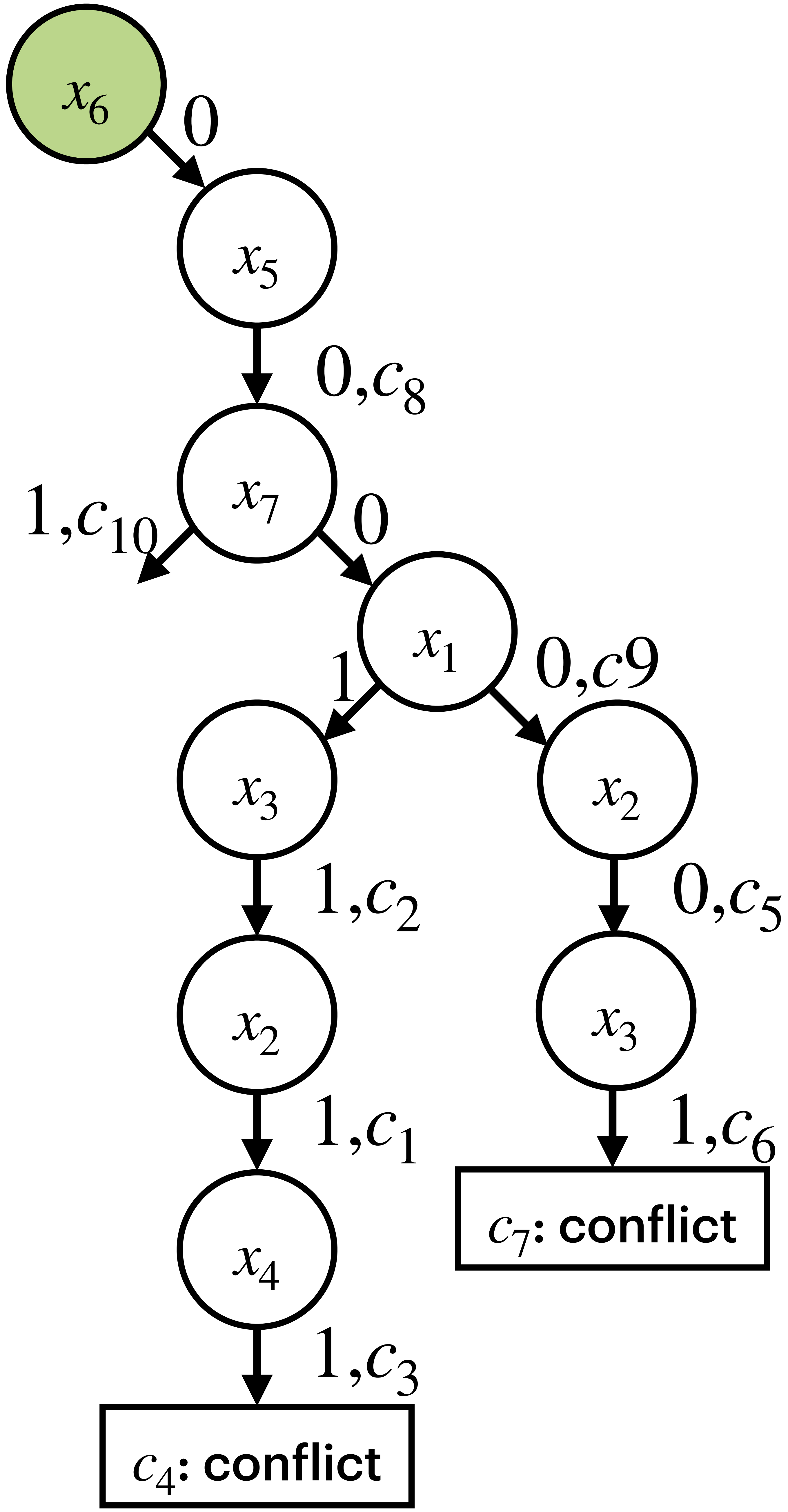
$$c_7 = (x_2 \vee \neg x_3 \vee x_7)$$

$$c_8 = (x_6 \vee \neg x_5)$$

$$c_9 = x_6 \vee \neg x_1$$

$$c_{10} = x_7 \vee x_6$$



$$c_1 = (\neg x_1 \vee x_2)$$

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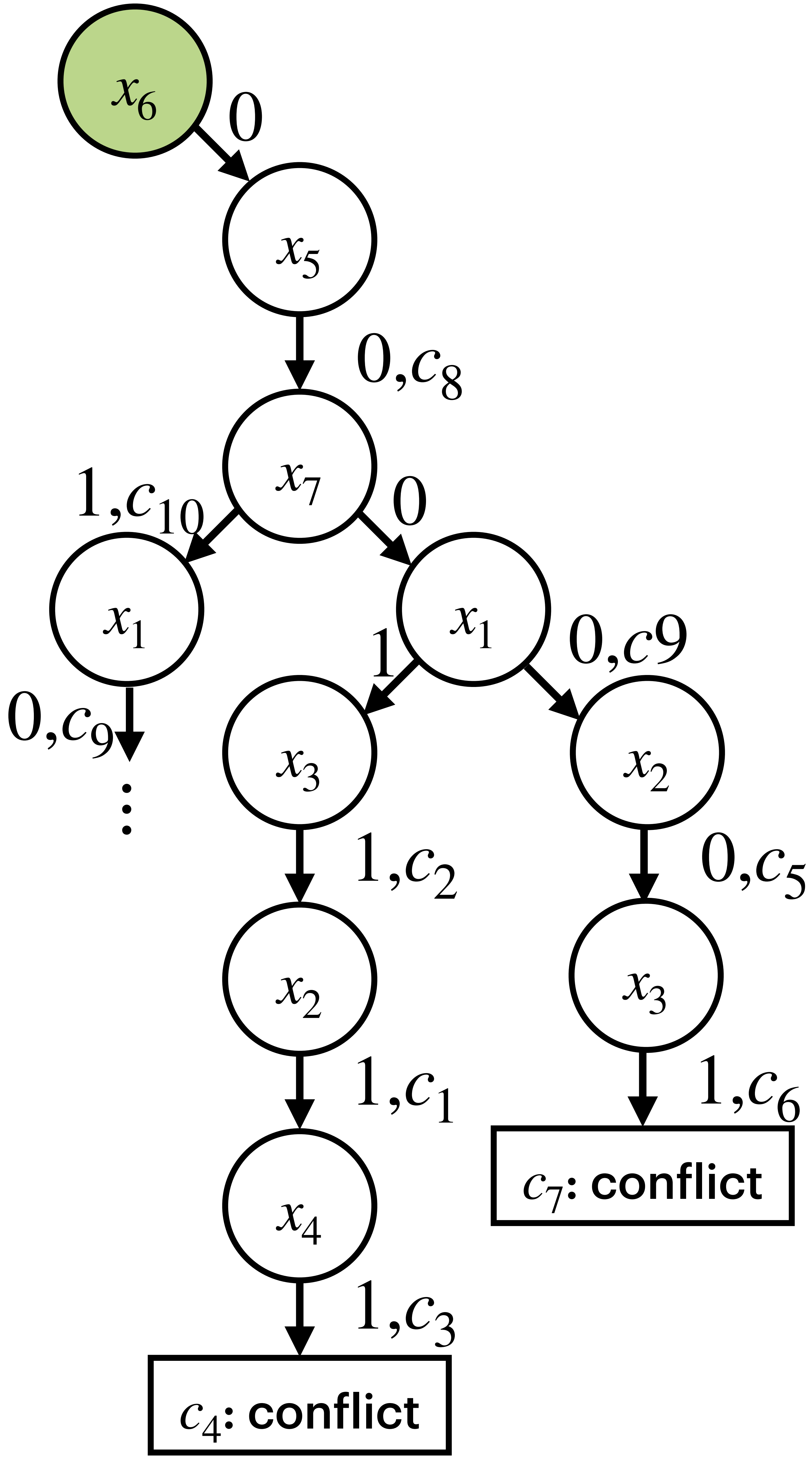
$$c_9 = x_6 \vee \neg x_1$$

$$c_{10} = x_7 \vee x_6$$



$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_5)$$

$$c_3 = (\neg x_2 \vee x_4)$$

$$c_4 = (\neg x_3 \vee \neg x_4)$$

$$c_5 = (x_1 \vee x_5 \vee \neg x_2)$$

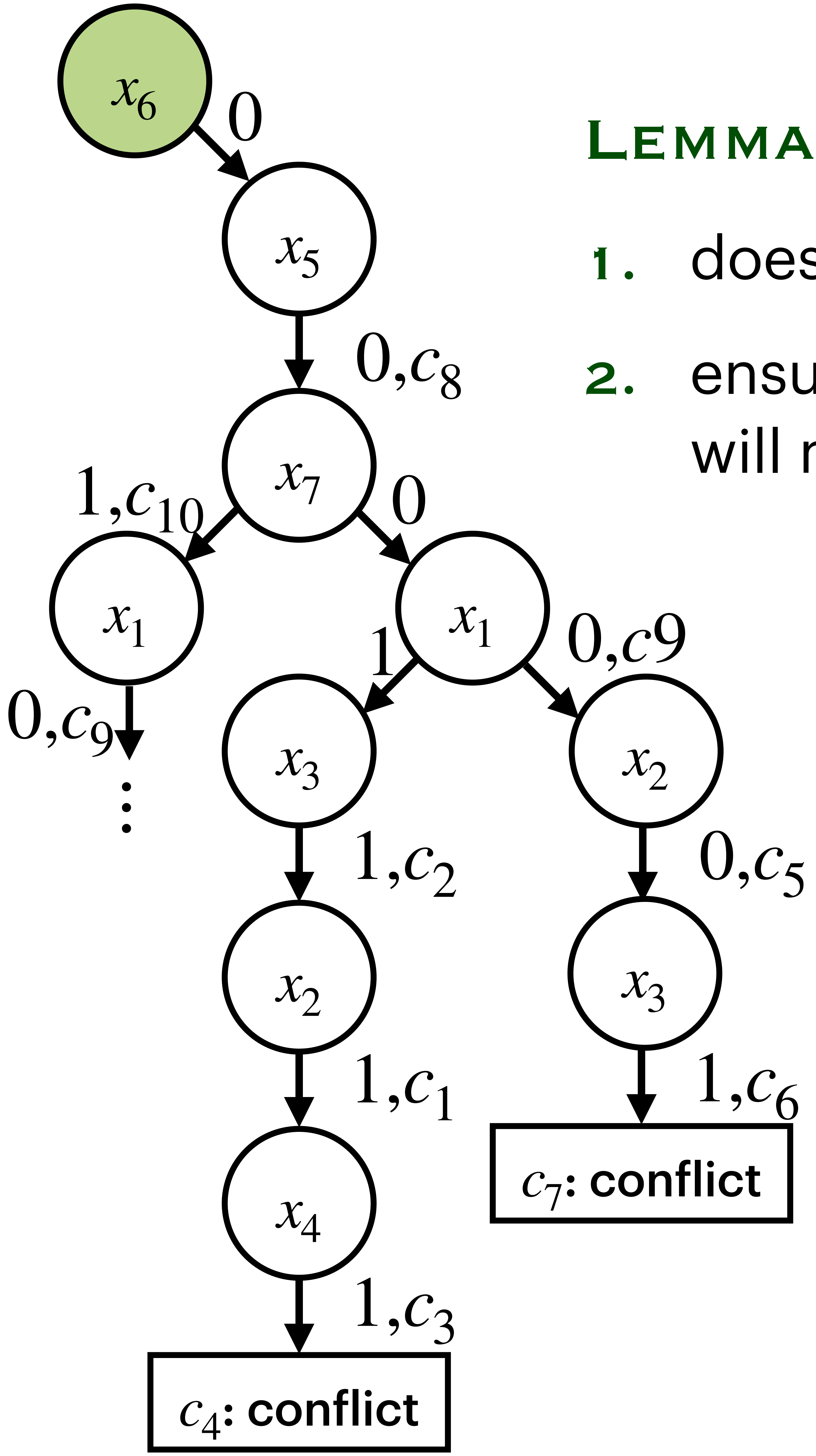
$$c_6 = (x_2 \vee x_3)$$

$$c_7 = (x_2 \vee \neg x_3 \vee x_7)$$

$$c_8 = (x_6 \vee \neg x_5)$$

$$c_9 = x_6 \vee \neg x_1$$

$$c_{10} = x_7 \vee x_6$$



LEMMA: adding conflict clauses

1. does not change satisfying assignments
2. ensures conflicting partial assignments will not be retried.

Davis–Putnam–Logemann–Loveland (DPLL) Algorithm

Input: CNF f, and partial assignment m

If f is true under m, return m.

If f is false under m, return $\bot$.

If $\exists$ unit literal p under m, then return $DPLL\left(f, m[p \to 1]\right)$.

If $\exists$ unit literal $\neg p$ under m, then return $DPLL\left(f, m[p \to 0]\right)$.

Chose an unassigned variable a, and assign it $b \in \{0,1\}$.

If $DPLL\left(f, m[a \to b]\right) = SAT$, return $m[a \to b]$

Else, return $DPLL\left(f, m[a \to 1 - b]\right)$

Conflict Driven Clause Learning (CDCL) Algorithm

Input: CNF f, and partial assignment m

If f is true under m, return m.

If f is false under m, return $\bot$.

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Conflict Driven Clause Learning (CDCL) Algorithm

Input: CNF f, and partial assignment m

If f is true under m, return m.

If f is false under m, return $\bot$.

UnitPropagation(m, f)

Chose an unassigned variable a, and assign it $b \in \{0,1\}$.

If $DPLL\left(f, m[a \rightarrow b]\right) = SAT$, return $m[a \rightarrow b]$

Else, return $DPLL\left(f, m[a \rightarrow 1 - b]\right)$

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If f is true under m, return m.

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Conflict Driven Clause Learning (CDCL) Algorithm

Input: CNF f, and partial assignment m

If f is true under m, return m.

If f is false under m:

Let $c =$ Analyse conflict(m, f); $f := f \cup \{c\}$.

UnitPropagation(m, f)

Chose an unassigned variable a, and assign it $b \in \{0,1\}$.

If $DPLL\left(f, m[a \to b]\right) = SAT$, return $m[a \to b]$

Else, return $DPLL\left(f, m[a \to 1 - b]\right)$

Conflict Driven Clause Learning (CDCL) Algorithm

Input: CNF f

$m := 0;\ dl := 0;\ dstack := \{\}$

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While $(m \not\models f)$ or (m is partial):

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If m is partial:

(m, dl) = Decide(m, f).

m = UnitPropagation(m, f).

Conflict Driven Clause Learning (CDCL) Algorithm

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$m =$ UnitPropagation(m, f).

If m is partial:

$(m, dl) =$ Decide(m, f).

$m =$ UnitPropagation(m, f).

DLIS (Dynamic Largest Individual Sum): Chooses a literal l with maximal occurrences in f.

DLCS (Dynamic Largest Clause Sum): Chooses a literal l with maximal occurrences of l and $\neg l$ in f.

MOM (Maximum Occurrence in Minimal Size Clauses): Let k be the shortest size clause in f. Choose a literal l with maximal occurrences of l and $\neg l$ in k-sized clauses of f.

Time to Code!

Greedy Search?

Start with any assignment.

Flip the variable that minimises the number of unsatisfied clauses.

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Start with any assignment.

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When does this not work?

Randomized Greedy Search?

Start with any assignment.

With probability p,
Flip the variable that minimises the number of unsatisfied clauses.

With probability $(1 - p)$,
Flip any variable in an unsatisfied clause.

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Random Walk

WalkSAT Algorithm:

Start with any assignment.

With probability p,
Flip the variable that minimises the number of unsatisfied clauses.

With probability $(1 - p)$,
Flip any variable in an unsatisfied clause.

WalkSAT Algorithm:

Start with any assignment.

With probability p,
Flip the variable that minimises the number of unsatisfied clauses.

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WalkSAT Algorithm:

Start with any assignment.

If there is a variable that can be flipped such that it does not turn any satisfied clauses into unsatisfied, flip it. Otherwise:

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Flip the variable that minimises the number of unsatisfied clauses.

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Flip any variable in an unsatisfied clause.

Hill Climbing + Random Walk

An Extension:

$$p := A \mid p \wedge p \mid p \vee p \mid \neg p$$

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$$p := A \mid p \wedge p \mid p \vee p \mid \neg p$$

$$A := (e = e)$$

An Extension:

$$p := A \mid p \wedge p \mid p \vee p \mid \neg p$$

$$A := (e = e)$$

$$e \in \mathbb{R} \cup V \qquad e := e \backsim e$$

$$\backsim \; := + \mid -$$

An Example:

$$p : \neg(x = 0) \wedge \big((x + y = 3.5) \vee (y - x = 2)\big)$$

Are there (x, y) such that p can be satisfied?

An Example:

$$p : \neg(x = 0) \wedge \big((x + y = 3.5) \vee (y - x = 2)\big)$$

$$A \qquad\qquad\qquad B \qquad\qquad\qquad C$$

Are there (x, y) such that p can be satisfied?

An Example:

$$p : \neg(x = 0) \wedge \big((x + y = 3.5) \vee (y - x = 2)\big)$$

$$p' : \neg A \wedge (B \vee C)$$

Are there (x, y) such that p can be satisfied?

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An Example:

$$p : \neg(x = 0) \wedge \big((x + y = 3.5) \vee (y - x = 2)\big)$$

$$p' : \neg A \wedge (B \vee C)$$

Are there (x, y) such that p can be satisfied?

$$\{A : 0, B : 1, C : 0\}$$

$$\{A : 0, B : 0, C : 1\}$$

$$\{A : 0, B : 1, C : 1\}$$

An Example:

$$p : \neg(x = 0) \wedge \big((x + y = 3.5) \vee (y - x = 2)\big)$$

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Are there (x, y) such that p can be satisfied?

$$\big\{(x, y) \,|\, x \neq 0,\, x + y = 3.5,\, y - x = 2\big\}$$

$$\{A : 0,\, B : 1,\, C : 1\}$$

An Example:

$$p : \neg(x = 0) \wedge \big((x + y = 3.5) \vee (y - x = 2)\big)$$

$$p' : \neg A \wedge (B \vee C)$$

Are there (x, y) such that p can be satisfied?

$$\big\{(x, y) \,|\, x \neq 0,\ x + y = 3.5,\ y - x = 2\big\}$$

$$\big\{(0.75, 2.75)\big\}$$

$$\{A : 0,\ B : 1,\ C : 1\}$$

An Example:

$$p : \neg(x = 0) \wedge \big((x + y = 3.5) \vee (y + x = 2)\big)$$

$$p' : \neg A \wedge (B \vee C)$$

Are there (x, y) such that p can be satisfied?

$$\big\{(x, y) \,|\, x \neq 0,\ x + y = 3.5,\ y + x = 2\big\}$$

$$\{A : 0,\ B : 1,\ C : 1\}$$

An Example:

$$p : \neg(x = 0) \land \big((x + y = 3.5) \lor (y + x = 2)\big)$$

$$p' : \neg A \land (B \lor C)$$

Are there (x, y) such that p can be satisfied?

$$\{A : 0,\ B : 1,\ C : 0\}$$

$$\{A : 0,\ B : 0,\ C : 1\}$$

~~$\{A : 0,\ B : 1,\ C : 1\}$~~

An Example:

$$p : \neg(x = 0) \wedge \big((x + y = 3.5) \vee (y + x = 2)\big)$$

$$p' : \neg A \wedge (B \vee C)$$

Are there (x, y) such that p can be satisfied?

$$\{A : 0,\ B : 1,\ C : 0\}$$

$$\{A : 0,\ B : 0,\ C : 1\}$$

$$\sout{\{A : 0,\ B : 1,\ C : 1\}}$$

Conflict Driven Clause Learning (CDCL) Algorithm

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If m is partial:

$(m, dl) =$ Decide(m, f).

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Why does this work?

$$p : \neg(x = 0) \wedge \big((x + y = 3.5) \vee (y - x = 2)\big)$$

$$p' : \neg A \wedge (B \vee C)$$

Are there (x, y) such that p can be satisfied?

$$\big\{(x, y) \,|\, x \neq 0,\, x + y = 3.5,\, y - x = 2\big\}$$

$$\big\{(0.75,2.75)\big\}$$

$$\{A : 0,\, B : 1,\, C : 1\}$$

Why does this work?

$$p : \neg(x = 0) \wedge \big((x + y = 3.5) \vee (y - x = 2)\big)$$

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Are there (x, y) such that p can be satisfied?

$$\big\{(x, y) \,|\, x \neq 0,\, x + y = 3.5,\, y - x = 2\big\}$$ Decidability of SLE

$$\big\{(0.75, 2.75)\big\}$$

$$\{A : 0,\, B : 1,\, C : 1\}$$

More Theories

$$p_1 : \big(y = g(y)\big) \wedge \big(f(x) = f(g(x))\big)$$

$$p_2 : (i \neq j) \rightarrow read(write(a, i, v), j) = read(a, j)$$

$$\big(read(A, x) = y\big) \wedge \big(f(x) = f(y)\big) \wedge \big(2x > y\big)$$

Theory of Equality Logic with Uninterpreted Functions

$$p = A \mid p \wedge p \mid p \vee p \mid p \rightarrow p \mid \neg p$$

$$A := e = e$$

$$e := f(e) \mid x$$

Theory of Equality Logic with Uninterpreted Functions

$$p = A \mid p \wedge p \mid p \vee p \mid p \rightarrow p \mid \neg p$$

$$A := e = e$$

$$e := f(e) \mid x$$

$$\forall x \,.\, (x = x)$$

$$\forall x, y \,.\, (x = y) \rightarrow (y = x)$$

$$\forall x, y, z \,.\, (x = y) \wedge (y = z) \rightarrow (x = z)$$

$$\forall x, y \,.\, (x = y) \rightarrow \big(f(x) = f(y)\big)$$

Theory of Equality Logic with Uninterpreted Functions

$$f(f(f(a))) = a \quad \land \quad f(f(f(f(f(a))))) = a \quad \land \quad \neg\big(f(a) = a\big)$$

$$f(f(f(a))) = a$$

Theory of Equality Logic with Uninterpreted Functions

$$f(f(f(a))) = a \quad \wedge \quad f(f(f(f(f(a))))) = a \quad \wedge \quad \neg\big(f(a) = a\big)$$

$$f(f(f(f(a)))) = f(a)$$

Theory of Equality Logic with Uninterpreted Functions

$$f(f(f(a))) = a \quad \land \quad f(f(f(f(f(a))))) = a \quad \land \quad \neg\big(f(a) = a\big)$$

$$f(f(f(f(f(a))))) = f(f(a))$$

Theory of Equality Logic with Uninterpreted Functions

$$f(f(f(a))) = a \quad \wedge \quad f(f(f(f(f(a))))) = a \quad \wedge \quad \neg\big(f(a) = a\big)$$

$$a = f(f(a))$$

Theory of Equality Logic with Uninterpreted Functions

$$f(f(f(a))) = a \quad \wedge \quad f(f(f(f(f(a))))) = a \quad \wedge \quad \neg\left(f(a) = a\right)$$

$$f(a) = f(f(f(a)))$$

Theory of Equality Logic with Uninterpreted Functions

$$f(f(f(a))) = a \quad \wedge \quad f(f(f(f(f(a))))) = a \quad \wedge \quad \neg\big(f(a) = a\big)$$

$$f(a) = a$$

Theory of Equality Logic with Uninterpreted Functions

$$f(f(f(a))) = a \quad \wedge \quad f(f(f(f(f(a))))) = a \quad \wedge \quad \neg\big(f(a) = a\big)$$

$$\bot$$

Theory of Equality Logic with Uninterpreted Functions

$$p = A \mid p \wedge p \mid p \vee p \mid p \rightarrow p \mid \neg p$$

$$A := e = e$$

$$e := f(e) \mid x$$

$$\forall x \,.\, (x = x)$$

$$\forall x, y \,.\, (x = y) \rightarrow (y = x)$$

$$\forall x, y, z \,.\, (x = y) \wedge (y = z) \rightarrow (x = z)$$

$$\forall x, y \,.\, (x = y) \rightarrow \big(f(x) = f(y)\big)$$

Theory of Equality Logic with Uninterpreted Functions

Design an algorithm to decide if a given statement in EUF is a theorem.

Theory of Equality Logic with Uninterpreted Functions

```
int a = f(x);
int b = g(a, y);
int a = h(b, z);
return a;
```

```
int a = h(g(f(x),y), z);
return a;
```

Are these two equivalent?

Examples

$$p_1 : \big(7 = g(7)\big) \wedge \big(f(x) = f(g(x))\big)$$

$$p_2 : (i \neq j) \rightarrow read(write(a, i, v), j) = read(a, j)$$

$$\big(read(A, x) = y\big) \wedge \big(f(x) = f(y)\big) \wedge \big(2x > y\big)$$

Examples

$$p_1 : \big(7 = g(7)\big) \wedge \big(f(x) = f(g(x))\big)$$

$$p_2 : (i \neq j) \rightarrow read(write(a, i, v), j) = read(a, j)$$

$$\big(read(A, x) = y\big) \wedge \big(f(x) = f(y)\big) \wedge \big(2x > y\big)$$

Not Always!

Nelson-Oppen Theory Combination

Convert F into $F_1 \wedge F_2 \wedge \ldots \wedge F_n$ such that F_i has only terms from theory T_i

$$read(x + y, A) = write(x, y, A)$$

Nelson-Oppen Theory Combination

Convert F into $F_1 \wedge F_2 \wedge \ldots \wedge F_n$ such that F_i has only terms from theory T_i

$$read(x + y, A) = write(x, y, A)$$

$$\big(read(z, A) = write(x, y, A)\big) \wedge (z = x + y)$$

Nelson-Oppen Theory Combination

Convert F into $F_1 \wedge F_2 \wedge \ldots \wedge F_n$ such that F_i has only terms from theory T_i

$$0 \leq x \leq 1 \wedge \big(f(x) \neq f(0)\big) \wedge \big(f(x) \neq f(1)\big)$$

Nelson-Oppen Theory Combination

Convert F into $F_1 \wedge F_2 \wedge \ldots \wedge F_n$ such that F_i has only terms from theory T_i

$$0 \le x \le 1 \wedge \big(f(x) \neq f(0)\big) \wedge \big(f(x) \neq f(1)\big)$$

$$0 \le x \le 1 \wedge y = 0 \wedge z = 1 \wedge \big(f(x) \neq f(y)\big) \wedge \big(f(x) \neq f(z)\big)$$

Nelson-Oppen Theory Combination

Convert F into $F_1 \wedge F_2 \wedge \ldots \wedge F_n$ such that F_i has only terms from theory T_i

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Convert F into $F_1 \wedge F_2 \wedge \ldots \wedge F_n$ such that F_i has only terms from theory T_i

Let DP_i be a decision procedure for T_i. If $DP_i(F_i)$ returns $\bot$, then F is unsatisfiable.

Nelson-Oppen Theory Combination

Convert F into $F_1 \wedge F_2 \wedge \ldots \wedge F_n$ such that F_i has only terms from theory T_i

Let DP_i be a decision procedure for T_i. If $DP_i(F_i)$ returns $\perp$, then F is unsatisfiable.

If all $DP_i(F_i)$ return $\top$, then F may still be unsatisfiable.

Nelson-Oppen Theory Combination

Nelson-Oppen Theory Combination

$$F[\sim] := \bigwedge \{t = s \mid t \sim s \text{ and } t, s \in S\} \wedge \bigwedge \{t \neq s \mid t \nsim s \text{ and } t, s \in S\}$$

Nelson-Oppen Theory Combination

Let T_1 and T_2 be two theories with disjoint signature.
Let F be a conjunction of literals for theory $C\left(T_1 \cup T_2\right)$.

1. Convert F into $F_1 \wedge F_2$.
2. Guess an equivalence relation $\sim$ over $vars\left(F_1\right) \cap vars\left(F_2\right)$.
3. Check $DP_1\left(F_1 \wedge F[\sim]\right)$.
4. Check $DP_2\left(F_2 \wedge F[\sim]\right)$.

Nelson-Oppen Theory Combination

$$0 \le x \le 1 \wedge \big(f(x) \neq f(0)\big) \wedge \big(f(x) \neq f(1)\big)$$

Nelson-Oppen Theory Combination

$$0 \leq x \leq 1 \land y = 0 \land z = 1 \land \big(f(x) \neq f(y)\big) \land \big(f(x) \neq f(z)\big)$$

Nelson-Oppen Theory Combination

$$0 \leq x \leq 1 \wedge y = 0 \wedge z = 1 \wedge \big(f(x) \neq f(y)\big) \wedge \big(f(x) \neq f(z)\big)$$

1. $x = y \quad \wedge \quad y = z \quad \wedge \quad z = x$
2. $x \neq y \quad \wedge \quad y \neq z \quad \wedge \quad z = x$
3. $x = y \quad \wedge \quad y \neq z \quad \wedge \quad z \neq x$
4. $x \neq y \quad \wedge \quad y = z \quad \wedge \quad z \neq x$
5. $x \neq y \quad \wedge \quad y \neq z \quad \wedge \quad z \neq x$

Nelson-Oppen Theory Combination

$$0 \leq x \leq 1 \land y = 0 \land z = 1 \land \big(f(x) \neq f(y)\big) \land \big(f(x) \neq f(z)\big)$$

1. $x = y \quad \land \quad y = z \quad \land \quad z = x$ ✗
2. $x \neq y \quad \land \quad y \neq z \quad \land \quad z = x$
3. $x = y \quad \land \quad y \neq z \quad \land \quad z \neq x$
4. $x \neq y \quad \land \quad y = z \quad \land \quad z \neq x$
5. $x \neq y \quad \land \quad y \neq z \quad \land \quad z \neq x$

Nelson-Oppen Theory Combination

$$0 \leq x \leq 1 \land y = 0 \land z = 1 \land \big(f(x) \neq f(y)\big) \land \big(f(x) \neq f(z)\big)$$

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Nelson-Oppen Theory Combination

$$0 \leq x \leq 1 \land y = 0 \land z = 1 \land \big(f(x) \neq f(y)\big) \land \big(f(x) \neq f(z)\big)$$

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Nelson-Oppen Theory Combination

$$0 \leq x \leq 1 \land y = 0 \land z = 1 \land \big(f(x) \neq f(y)\big) \land \big(f(x) \neq f(z)\big)$$

1. $x = y \quad \land \quad y = z \quad \land \quad z = x$ ✗
2. $x \neq y \quad \land \quad y \neq z \quad \land \quad z = x$ ✗
3. $x = y \quad \land \quad y \neq z \quad \land \quad z \neq x$ ✗
4. $x \neq y \quad \land \quad y = z \quad \land \quad z \neq x$ ✗
5. $x \neq y \quad \land \quad y \neq z \quad \land \quad z \neq x$

Nelson-Oppen Theory Combination

$$0 \leq x \leq 1 \land y = 0 \land z = 1 \land \big(f(x) \neq f(y)\big) \land \big(f(x) \neq f(z)\big)$$

1. $x = y \quad \land \quad y = z \quad \land \quad z = x$ ✗
2. $x \neq y \quad \land \quad y \neq z \quad \land \quad z = x$ ✗
3. $x = y \quad \land \quad y \neq z \quad \land \quad z \neq x$ ✗
4. $x \neq y \quad \land \quad y = z \quad \land \quad z \neq x$ ✗
5. $x \neq y \quad \land \quad y \neq z \quad \land \quad z \neq x$?

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4. Check $DP_2\left(F_2 \wedge F[\sim]\right)$.

Time to Code!


Pre-doctoral Research Workshop
January 9-11, 2025